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QUALITATIVE STUDY FOR A COUPLED SYSTEM OF DIFFERENTIAL EQUATION ON THE REAL HALF-LINE

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ABSTRACT. This research paper focuses on investigating the solvability of the qualitative study for a coupled system of differential equation on the real half-line by applying Darboe's fixed point Theorem and the technique of the measure of noncompactness (MNC). This study has been located in space $BC(R_+)$. Furthermore, we prove the asymptotic stability of the solution of our problem, we introduce the idea of dependency of the solutions on some data. Additionally, we delve into the study of Hyers-Ulam stability. Finally, we present an example to support our findings.

keywords: Differential equation; existence of solution; Hyers–Ulam stability; dependency; coupled system
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1. Introduction

The study of differential equations has received much attention over the last 30 years or so. For papers studying such kind of problems (see [14, 15, 38, 39]) and the references therein.
It is known that the nonlinear initial value problems create an important branch of nonlinear analysis and have numerous applications in describing of miscellaneous real world problems. Such kind of these equations have been considered in numerous papers see [5] and references therein.
The technique associated with MNC in the Banach space $BC(R_+)$ have been successfully used by J. Banaś (see [5, 35, 2]) to prove the existence of asymptotically stable solutions for some functional equation (see [12, 13]).

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The authors in [31] discussed the problem

$$\frac{dx}{dt} = f(t, x(t)),\ t \in (0, \infty),$$

with the nonlocal integral condition

$$x(\tau) + \int_0^\tau g(s, x(s))ds = x_0,\ \tau \geq 0$$

in an unbounded interval. Also, they studied the solvability of these problem using the technique of MNC in an infinite interval and also discussed the asymptotically stable and dependency.

Here we are concerning with the coupled system of initial value problem of the functional differential equations

$$\frac{dx}{dt} = f_1(t, y(\phi_1(t))),\ x(0) = x_0,\ t \in (0, \infty) \tag{1.1}$$

and

$$\frac{dy}{dt} = f_2(t, x(\phi_2(t))),\ y(0) = y_0,\ t \in (0, \infty). \tag{1.2}$$

Our aim here is to establish the solvability of the solution $(x,\ y) \in BC(R_+) \times BC(R_+)$ of the problem (1.1)-(1.2). The main tools in our study is applying Darbo's fixed point Theorem [19] and MNC technique. Furthermore, the asymptotic stability and dependency of $(x,\ y) \in BC(R_+) \times BC(R_+)$ on the initial data $x_0,\ y_0$ and on the functions f_i and $\phi_i,\ i\ =\ 1,\ 2$ has been studied. The Hyers – Ulam stability of the problem (1.1)-(1.2) will be studied. Finally, we give an example illustrate our results.

The main tool in our work are the measure of noncompactness and Darbo fixed point Theorem [19].
Let $BC(R_+)$ be the class of all bounded and continuous functions in R_+, with the standard norm

$$\|x\|_{BC(R_+)}\ = \|x\|^* = \sup_{t \in R_+} |x(t)|$$

and $E\ = BC(R_+) \times BC(R_+)$ be the Banach space with the norm

$$\|(x,\ y)\|_E =\ max\ \{\ \|x\|^*,\ \|y\|^*\ \}.$$

Now [25, 28], let $E = BC(R_+) \times BC(R_+),\ X,\ Y \subset BC(R_+)$ and

$$U = \{\ u \in U : u = (x, y), x \in X, y \in Y\ \} = X \times Y.$$

Then, we can introduce the following:

$$\omega^T(x, \epsilon) = sup\ \{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leq \epsilon\ \},$$
$$\omega^T(y, \epsilon) = sup\ \{|y(t) - y(s)| : t, s \in [0, T], |t - s| \leq \epsilon\ \}$$

and

$$\omega^T(u, \epsilon)\ =\ max\ \{\ \omega^T(x, \epsilon),\ \omega^T(y, \epsilon)\ \},$$

then

$$\omega^T(U, \epsilon)\ =\ sup\ \omega^T(u, \epsilon) : u \in U,$$
$$\omega_0^T(U) = \lim_{\epsilon \to 0} \omega^T(U, \epsilon),\ \omega_0(U) = \lim_{T \to \infty} \omega_0^T(U).$$

Also,

$$\begin{aligned} \omega(U) &= \omega(X \times Y) = max \{ \omega(X), \omega(Y) \}, \\ diam(U) &= diam(X \times Y) = max \{ diam(X), diam(Y) \}, \\ \lim_{t\to\infty} sup\ diam(U) &= max \{ \lim_{t\to\infty} sup\ diam(X), \lim_{t\to\infty} sup\ diam(Y) \} \end{aligned}$$

and

$$\mu(U) = \omega_0(U) + \lim_{t\to\infty} sup\ diamU(t). \quad (1.3)$$

Finally, we state the Darbo fixed point Theorem [19].
The following Theorem will be needed.

Theorem 1.1. *Let Q be nonempty bounded closed convex subset of the space E and let $F : Q \to Q$ be a continuous operator such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset X of Q, where $k \in [0, 1)$ is a constant. Then F has a fixed point in the set Q.*

2. Existence of solution

Consider now the problem (1.1)-(1.2) under the following assumptions:

(i) $\phi_i : R_+ \to R_+,\ i = 1, 2, \phi_i(t) \leq t$ are continuous and increasing.

(ii) $f_i : R_+ \times R \to R,\ i = 1, 2$ are continuous in $t \in R_+, \forall\ x, y \in R$ and satisfies Lipschitz condition

$$|f_i(t, x) - f_i(t, y)| \leq b_i(t)|x - y|\ \forall\ t \in R_+,\ x, y \in R, \quad (2.1)$$

where b_i is integrable $\int_0^t b_i(s)ds \leq \int_0^T b_i(s)ds \leq b_i$ and

$$\lim_{t\to\infty} \int_0^t b_i(s)ds = 0,\ \sup_{t\in R_+} \int_0^t b_i(s)ds < b_i^*,\ i = 1, 2.$$

(iii) $b^* < 1$, where $b^* = max \{ b_1^*, b_2^* \}$.

From equation (2.1), we have

$$|f_i(t, x)| - |f_i(t, 0)| \leq |f_i(t, x) - f_i(t, 0)| \leq b_i(t)|x|,$$

$$|f_i(t, x)| \leq |f_i(t, 0)| + b_i(t)|x|$$

and

$$|f_i(t, x)| \leq |m_i(t)| + b_i(t)|x|,$$

where

$$|m_i(t)| = |f_i(t, 0)| \in BC(R_+) < \infty,\ \lim_{t\to\infty} \int_0^t |m_i(s)|ds = 0 \ and\ \sup_{t\in R_+} \int_0^t |m_i(s)|ds < m_i^*.$$

Now, the following lemma.

Lemma 2.1. *The coupled system of the functional differential equations (1.1)-(1.2) is equivalent to the functional integral equations*

$$x(t) = x_0 + \int_0^t f_1(s, y(\phi_1(s)))ds,\ t \geq 0 \quad (2.2)$$

and

$$y(t) = y_0 + \int_0^t f_2(s, x(\phi_2(s)))ds,\ t \geq 0. \quad (2.3)$$

Proof. Let $(x,y) \in BC(R_+) \times BC(R_+)$ be a solution of the problem (1.1)-(1.2), then by integrability we get

$$\begin{aligned} x(t) - x(0) &= \int_0^t f_1(s, y(\phi_1(s)))ds \\ x(t) &= x(0) + \int_0^t f_1(s, y(\phi_1(s)))ds \end{aligned}$$

and

$$y(t) = y(0) + \int_0^t f_2(s, x(\phi_2(s)))ds.$$

Substituting $x(0) = x_0$ and $y(0) = y_0$, we obtain (2.2) and (2.3).
Conversely, let $(x,\ y) \in BC(R_+) \times BC(R_+)$ be a solution of (2.2)-(2.3).
Differentiation (2.2)-(2.3), we obtain

$$\frac{dx}{dt} = f_1(t, y(\phi_1(t)))$$

and

$$\frac{dy}{dt} = f_2(t, x(\phi_2(t))).$$

Let $t = 0$, we have

$$x(0) = x_0 \ and \ y(0) = y_0.$$

Now, we have the following existences theorem.

Theorem 2.2. *Assume that $(i) - (iii)$ be satisfied, then the coupled system (1.1)-(1.2) has at least one solution $(x,\ y) \in BC(R_+) \times BC(R_+)$.*

Proof. Define the set

$$\begin{aligned} Q_r &= \{\ (x,\ y) \in BC(R_+) \times BC(R_+) : \|x\|^* \le r_2,\ \|y\|^* \le r_1,\ max\ \{\ r_1,\ r_2\ \} \le r\}, \\ r &= \frac{a + m^*}{1 - b^*},\ where\ a = max\ \{\ |x_0|,\ |y_0|\ \}\ and\ m^* = max\ \{\ m_1^*,\ m_2^*\ \}. \end{aligned}$$

Let $F_1,\ F_2$ be defined on $BC(R_+)$ by

$$F_1 y(t) = x_0 + \int_0^t f_1(s, y(\phi_1(s)))ds,\ t \in R_+, \tag{2.4}$$

$$F_2 x(t) = y_0 + \int_0^t f_2(s, x(\phi_2(s)))ds,\ t \in R_+ \tag{2.5}$$

and F is given by

$$F(x,\ y)(t) = (F_1 y(t),\ F_2 x(t)).$$

Now, let $(x,\ y) \in Q_r$, then

$$\begin{aligned} |F_1 y(t)| &= \left| x_0 + \int_0^t f_1(s, y(\phi_1(s)))ds \right| \\ &\leq |x_0| + \int_0^t |f_1(s, y(\phi_1(s)))|ds \\ &\leq |x_0| + \int_0^t |m_1(s)|ds + \int_0^t |b_1(s)||y(\phi_1(s))|ds \\ &\leq |x_0| + m_1^* + \|y\|^* \int_0^t |b_1(s)|ds \\ &\leq |x_0| + m_1^* + r_1\ b_1^*, \end{aligned}$$

then

$$\|F_1 y\|^* \leq |x_0| + m_1^* + r_1\ b_1^* = r_1,\ r_1 = \frac{|x_0| + m_1^*}{1 - b_1^*}.$$

Similarly, we obtain

$$\|F_2 x\|^* \leq |y_0| + m_2^* + r_2\ b_2^* = r_2,\ r_2 = \frac{|y_0| + m_2^*}{1 - b_2^*},$$

then

$$\|F(x,\ y)\| = \|(F_1 y,\ F_2 x)\| = max\ \{\ \|F_1 y\|^*,\ \|F_2 x\|^*\ \} \leq r.$$

Hence the operator $F : Q_r \longrightarrow Q_r$.
Next, we prove that F is continuous on the ball Q_r.
Now, let $\delta > 0$ be given and take $(x_1,\ y_1),\ (x_2,\ y_2) \in U \subset Q_r$, such that $\|x_2 - x_1\|^* \leq \delta$ and $\|y_2 - y_1\|^* \leq \delta$, then

$$\begin{aligned} |F_1 y_2(t) - F_1 y_1(t)| &= \left| x_0 + \int_0^t f_1(s, y_2(\phi_1(s)))ds - x_0 - \int_0^t f_1(s, y_1(\phi_1(s)))ds \right| \\ &\leq \int_0^t \left| f_1(s, y_2(\phi_1(s))) - f_1(s, y_1(\phi_1(s))) \right| ds \\ &\leq \int_0^t |b_1(s)||y_2(\phi_1(s)) - y_1(\phi_1(s))|ds. \end{aligned} \tag{2.6}$$

(i) Choose $T > 0$ such that $t \geq T$, then

$$\begin{aligned} \|F_1 y_2 - F_1 y_1\|^* &\leq \|y_2 - y_1\|^* \int_0^t |b_1(s)|\ ds \\ &\leq \delta\ b_1^* = \epsilon. \end{aligned}$$

(ii) Also, for $T > 0$, $t \in [0, T]$ then from (2.6), we get

$$\begin{aligned} \|F_1 y_2 - F_1 y_1\|^* &\leq \|y_2 - y_1\|^* \int_0^t |b_1(s)|\ ds \\ &\leq \|y_2 - y_1\|^* \int_0^T |b_1(s)|\ ds \\ &\leq \delta\ b_1 = \epsilon_1. \end{aligned}$$

We can deduce that the operator F_1 is a continuous operator and by the same way we can prove F_2 is also a continuous operator.

Hence the operator $F(x,\ y) = (F_1 y,\ F_2 x) : Q_r \longrightarrow Q_r$ is continuous.
Now, let $U = X \times Y$ be nonempty subset of Q_r. Fix $\epsilon > 0$ and choose $y \in Y$ and $t_1,\ t_2 \in R_+$ such that $|t_2 - t_1| \leq \delta$, then

$$\begin{aligned} |F_1 y(t_2) - F_1 y(t_1)| &= \left| x_0 + \int_0^{t_2} f_1(s, y(\phi_1(s)))ds - x_0 - \int_0^{t_1} f_1(s, y(\phi_1(s)))ds \right| \\ &\leq \int_{t_1}^{t_2} |f_1(s, y(\phi_1(s)))|ds. \end{aligned}$$

Now, let $t_1,\ t_2 \in [0, T]$, $|t_2 - t_1| < \delta$, then we deduce that

$$\omega^T(F_1 y, \epsilon) \leq \int_{t_1}^{t_2} |f_1(s, y(\phi_1(s)))|ds < \epsilon$$

$$\omega_0^T(F_1 Y) \leq 0$$

and as $T \to \infty$

$$\omega_0(F_1 Y) = 0.$$

Similarly, we can deduce that

$$\omega_0(F_2 X) = 0,$$

then

$$\omega_0(FU) = max\{\omega_0(F_1 Y),\ \omega_0(F_2 X)\ \} = 0.$$

Hence

$$\omega_0(FU) = 0. \tag{2.7}$$

Moreover, for any $v_1 = (x_1,\ y_1)$, $v_2 = (x_2,\ y_2) \in U \subset Q_r$ and fixed $t \geq 0$, then from (2.2) and (2.3) we get

$$\begin{aligned} |x_2(t) - x_1(t)| &\leq \int_0^t f_1(s,\ y_2(\phi_1(s))) - f_1(s,\ y_1(\phi_1(s))) \\ &\leq \int_0^t |b_1(s)||y_2(\phi_1(s)) - y_1(\phi_1(s))|ds \leq 2\ r\ b_1^* \end{aligned}$$

and

$$\begin{aligned} \sup_{x_2, x_1 \in X} |x_1(t) - x_2(t)| &\leq 2\ r\ b_1^* \\ 0 \leq diam\ X(t) &\leq 2\ r\ b_1^*, \end{aligned}$$

then

$$\lim_{t \to \infty} diam\ X(t)\ be\ exist\ and \in [0,\ 2\ r\ b_1^*].$$

Similarly,

$$\lim_{t \to \infty} diam\ Y(t)\ be\ exist\ and \in [0,\ 2\ r\ b_2^*].$$

Now, from (2.4) and (2.5), we have

$$\begin{aligned}
|F_1y_2(t) - F_1y_1(t)| &\leq \int_0^t |b_1(s)||y_2(\phi_1(s)) - y_1(\phi_1(s))|ds \\
&\leq \int_0^t |b_1(s)| \sup_{y_1,y_2\in Y} |y_2(\phi_1(s)) - y_1(\phi_1(s))|ds \\
&\leq \int_0^t |b_1(s)|\ diam\ Y(s)ds \\
&\leq \int_0^t \Big(|b_1(s)| \lim_{s\to\infty} diam\ Y(s) + \epsilon\Big)ds \\
&\leq \Big(\lim_{t\to\infty} diam\ Y(t) + \epsilon^*\Big). \int_0^t |b_1(s)|ds \\
&\leq \Big(\lim_{t\to\infty} diam\ Y(t) + \epsilon^*\Big)b_1^* \\
&\leq \Big(\lim_{t\to\infty} diam\ Y(t)\Big)b_1^* + \epsilon^* b_1^* \\
&\leq \Big(\lim_{t\to\infty} diam\ Y(t)\Big)b_1^* + \epsilon_1,
\end{aligned}$$

then

$$diam\ F_1Y(t) \quad \leq \quad b_1^* \lim_{t\to\infty} diamY(t).$$

Hence

$$\lim_{t\to\infty} sup\ diam\ F_1\ Y(t) \quad \leq \quad b_1^* \lim_{t\to\infty} sup\ diamY(t).$$

Similarly, we can deduce that

$$\lim_{t\to\infty} sup\ diam\ F_2\ X(t) \quad \leq \quad b_2^* \lim_{t\to\infty} sup\ diam\ X(t).$$

Hence

$$diam\ (F_1Y,\ F_2X)(t) \quad = \quad max\ \{\ diam\ (F_1\ Y(t)),\ diam\ (F_2\ X(t))\ \}$$

then

$$\lim_{t\to\infty} sup\ diam\ (F_1Y,\ F_2X)(t)\ =\ b^*$$

and

$$\lim_{t\to\infty} sup\ diam\ F\ U(t)\ =\ b^*. \tag{2.8}$$

Now, from (2.7) and (2.8) and the definition of μ in (1.3), we obtain

$$\mu\ (FU)\ =\ b^*.$$

Then by Darbo fixed point Theorem [19] F has a fixed point $(x, y) \in U$, then the coupled system of functional integral equation (2.2)-(2.3) has at least one solution in the space $BC(R_+)$. Consequently the problem (1.1)-(1.2) has at least one solution in the space $BC(R_+)$.

3. Asymptotic stability

Theorem 3.1. *The solution $(x,\ y) \in BC(R_+) \times BC(R_+)$ of the coupled system (1.1)-(1.2) is asymptotically stable in the sense that for any $\epsilon > 0$, there exist $T(\epsilon) > 0$ and $r > 0$, such that, if any two solutions $(x, y), (x_1, y_1) \in U$ satisfy $\|(x, y) - (x_1, y_1)\| \leq \epsilon$, $t \geq T(\epsilon)$. This indicates that $|x(t) - x_1(t)| \leq \epsilon$ and $|y(t) - y_1(t)| \leq \epsilon$, $r \geq T(\epsilon)$.*

Proof. From Theorem 2.2, we have evaluated, we have

$$\begin{aligned}
\|y_2 - y_1\|^* &= \|F_1 y_2 - F_1 y_1\|^* \\
&= \left| \int_0^t f_1(s, y_2(\phi_1(s)))ds - \int_0^t f_1(s, y_1(\phi_1(s)))ds \right| \\
&\leq \int_0^t |f_1(s, y_2(\phi_1(s))) - f_1(s, y_1(\phi_1(s)))|ds \\
&\leq 2 \int_0^t |m_1(s)|ds + 2\ r_2 \int_0^t |b_1(s)|ds \\
&\leq 2\ \epsilon_1 + 2\ r_1\ \epsilon_2 = \frac{\epsilon}{2}.
\end{aligned}$$

Similarly,

$$\|x_2 - x_1\|^* \ \leq\ \frac{\epsilon}{2},$$

then

$$\begin{aligned}
\|(x_1,\ y_1) - (x_2,\ y_2)\| &= \|(F_1 y_1, F_2 x_1) - (F_1 y_2, F_2 x_2)\| \\
&= \|(F_1 y_1 - F_1 y_2, F_2 x_1 - F_2 x_2)\| \\
&= \|F_1 y_1 - F_1 y_2\|^* + \ \|F_2 x_1 - F_2 x_2\|^* \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Consequently, the coupled system (1.1)-(1.2) is asymptotically stable.

Corollary 3.2. Let the assumptions of theorem 2.2 be satisfied, then the solution of the problem (1.1)-(1.2) is unique.

4. Dependency

4.1. Dependency on the initial data $x_0,\ y_0$ and on the functions ϕ_i, i=1, 2.

Theorem 4.1. *Let the assumptions of Theorems 2.2 be satisfies, then the solution $(x,\ y) \in U$ of the coupled system (1.1)-(1.2) is asymptotically dependence on the initial data x_0 , y_0 and the functions ϕ_i , $i = 1,\ 2$ if $\forall\ \epsilon\ >\ 0$, $\exists\ \delta\ (\epsilon)$ such that*

$$max \left\{ \ |x_0 - x_0^*|,\ |y_0 - y_0^*|,\ |\phi_i - \phi_i^*| \ \right\} \ <\ \delta,$$

$$then\ \|(x, y) - (x_s, y_s)\| < \epsilon,$$

where x^* be a solution of

$$x^*(t) = x_0^* + \int_0^t f_1(s, y^*(\phi_1^*(s)))ds,\ t \geq 0$$

and y^* *be a solution of*

$$y^*(t) = y_0^* + \int_0^t f_2(s, x^*(\phi_2^*(s)))ds, \ t \geq 0.$$

Proof. Let $(x, y), (x^*, y^*) \in U$ be two solutions of the coupled system (2.2)-(2.3), then

$$\begin{aligned}
|x(t) - x^*(t)| &= \left| x_0 + \int_0^t f_1(s, y(\phi_1(s)))ds \right. \\
&- \left. x_0^* - \int_0^t f_1(s, y^*(\phi_1^*(s)))ds \right| \\
&\leq |x_0 - x_0^*| + \int_0^t |f_1(s, y(\phi_1(s))) - f_1(s, y^*(\phi_1^*(s)))|ds \\
&\leq \delta + \int_0^t |f_1(s, y(\phi_1(s))) - f_1(s, y^*(\phi_1(s)))|ds \\
&+ \int_0^t |f_1(s, y^*(\phi_1(s))) - f_1(s, y^*(\phi_1^*(s)))|ds,
\end{aligned}$$

then

$$\begin{aligned}
\|x - x^*\|^* &\leq \delta + b_1^* \|y - y^*\|^* + b_1^* |y^*(\phi_1(t)) - y^*(\phi_1^*(t))| \\
&\leq \delta + b_1^* \|y - y^*\|^* + b_1^* \epsilon^*.
\end{aligned}$$

Hence

$$\|x - x^*\|^* \leq \delta + b_1^* \|y - y^*\|^* + b_1^* \epsilon^*.$$

Similarly,

$$\begin{aligned}
|y(t) - y^*(t)| &\leq |y_0 - y_0^*| + \int_0^t |f_2(s, x(\phi_2(s))) - f_2(s, x^*(\phi_2(s)))|ds \\
&+ \int_0^t |f_2(s, x^*(\phi_2(s))) - f_2(s, x^*(\phi_2^*(s)))|ds,
\end{aligned}$$

then

$$\|y - y^*\|^* \leq \delta + b_2^* \|x - x^*\|^* + b_2^* \epsilon^*.$$

Hence

$$\begin{aligned}
\|x - x^*\|^* &\leq \delta + b_1^* (\delta + b_2^* \|x - x^*\|^* + b_2^* \epsilon^*) + b_1^* \epsilon^* \\
&\leq \frac{\delta + b_1^* \delta + b_1^* b_2^* \epsilon^* + b_1^* \epsilon^*}{1 - b_1^* b_2^*}
\end{aligned}$$

and

$$\|y - y^*\|^* \leq \frac{\delta + b_2^* \delta + b_1^* b_2^* \epsilon^* + b_2^* \epsilon^*}{1 - b_1^* b_2^*}.$$

Hence

$$max \ \{ \|x - x^*\|^*, \ \|y - y^*\|^* \} \leq \frac{\delta + b^* \delta + b^{*2} \epsilon^* + b^* \epsilon^*}{1 - b^{*2}} = \epsilon.$$

Since

$$\begin{aligned}
\|(x, y) - (x^*, y^*)\| &= \|(x - x^*), (y - y^*)\| \\
&= max \ \{ \|(x - x^*)\|^*, \ \|(y - y^*)\|^* \} < \epsilon,
\end{aligned}$$

then

$$\|(x,y)-(x^*,y^*)\| < \epsilon.$$

4.2. **Dependency on the functions f_i, i=1, 2.**

Theorem 4.2. *Let the assumptions of Theorems 2.2 be satisfies, then the solution $(x,\ y) \in U$ of the coupled system (1.1)-(1.2) is asymptotically dependence on the functions f_i, $i=1,\ 2$ if $\forall\ \epsilon\ >\ 0,\ \exists\ \delta\ (\epsilon)$ such that*

$$\int_0^t |f_i(s,x(s)) - f_i^*(s,x(s))|ds \ < \ \delta,$$

$$\textit{then } \|(x,y)-(x_s,y_s)\| < \epsilon,$$

where x^* be a solution of

$$x^*(t) = x_0 + \int_0^t f_1^*(s, y^*(\phi_1(s)))ds,\ t \geq 0$$

and y^* be a solution of

$$y^*(t) = y_0 + \int_0^t f_2^*(s, x^*(\phi_2(s)))ds,\ t \geq 0.$$

Proof. Let $(x,y),(x^*,y^*) \in U$ be two solutions of the coupled system (2.2)-(2.3), then

$$\begin{aligned}
|x(t)-x^*(t)| &= \left| x_0 + \int_0^t f_1(s,y(\phi_1(s)))ds \right. \\
&- \left. x_0 - \int_0^t f_1^*(s,y^*(\phi_1(s)))ds \right| \\
&\leq \int_0^t |f_1(s,y(\phi_1(s))) - f_1^*(s,y^*(\phi_1(s)))|ds \\
&\leq \int_0^t |f_1(s,y(\phi_1(s))) - f_1(s,y^*(\phi_1(s)))|ds \\
&+ \int_0^t |f_1(s,y^*(\phi_1(s))) - f_1^*(s,y^*(\phi_1(s)))|ds,
\end{aligned}$$

then

$$\|x-x^*\|^* \quad \leq \quad b_1^*\|y-y^*\|^* + \ \delta.$$

Similarly,

$$\begin{aligned}
|y(t)-y^*(t)| &\leq \int_0^t |f_2(s,x(\phi_2(s))) - f_2(s,x^*(\phi_2(s)))|ds \\
&+ \int_0^t |f_2(s,x^*(\phi_2(s))) - f_2^*(s,x^*(\phi_2(s)))|ds,
\end{aligned}$$

then

$$\|y-y^*\|^* \quad \leq \quad b_2^*\ \|x-x^*\|^* + \ \delta.$$

Hence

$$\begin{aligned}
\|x-x^*\|^* &\leq b_1^*\ (b_2^*\ \|x-x^*\|^* + \ \delta) + \delta \\
&\leq \frac{b_1^*\ \delta\ +\ \delta}{1-b_1^*\ b_2^*}
\end{aligned}$$

and

$$\|y - y^*\|^* \quad \leq \quad \frac{b_2^* \ \delta \ + \ \delta}{1 - b_1^* \ b_2^*}.$$

Hence

$$max \ \{ \ \|x - x^*\|^*, \ \|y - y^*\|^* \} \quad \leq \quad \frac{b^* \ \delta \ + \ \delta}{1 - b^{*2}} \ = \ \epsilon.$$

Since

$$\begin{aligned} \|(x, y) - (x^*, y^*)\| &= \|(x - x^*), (y - y^*)\| \\ &= max \ \{ \ \|(x - x^*)\|^*, \ \|(y - y^*)\|^* \} < \epsilon, \end{aligned}$$

then

$$\|(x, y) - (x^*, y^*)\| < \epsilon.$$

5. Hyers - Ulam stability

Definition 5.1. [27, 37, 29] Let the solution $(x, \ y) \in U$ of the coupled system (2.2)-(2.3) be exists, then the problem (1.1)-(1.2) is Hyers-Ulam stable if $\forall \epsilon > 0, \ \exists \ \delta(\epsilon)$ *such that* for any $\delta - approximate$ solution of the coupled system (2.2)-(2.3), then $(x_s, y_s) \in U$ satisfies,

$$max \left\{ \left| \frac{dx_s}{dt} - f_1(t, y_s(\phi_1(t))) \right|, \ \left| \frac{dy_s}{dt} - f_2(t, x_s(\phi_2(t))) \right| \right\} < \delta \ a(t) \quad (5.1)$$
$$implies \ \|(x, y) - (x_s, y_s)\| \ < \ \epsilon,$$

where $\sup_{t \in R_+} \int_0^t \ a(s) \ ds \ \leq \ k.$

Theorem 5.2. *Let the assumptions of Theorem 2.2 be satisfied, then the coupled system (2.2)-(2.3) is Hyers - Ulam stable.*

Proof. From (5.1), we have

$$\begin{aligned} -\delta \ a(t) &\leq \frac{dx_s}{dt} - f_1(t, y_s(\phi_1(t))) \leq \delta \ a(t) \\ -\delta^* = \ -\delta \int_0^t \ a(s)ds &\leq x_s(t) - x_s(0) + \int_0^t \ f_1(s, y_s(\phi_1(s)))ds \leq \delta \int_0^t \ a(s)ds = \delta^* \\ -\delta^* &\leq x_s(t) - x_0 + \int_0^t \ f_1(s, y_s(\phi_1(s)))ds \leq \ \delta^*. \end{aligned}$$

Similarly,

$$\begin{aligned} -\delta \ a(t) &\leq \frac{dy_s}{dt} - f_2(t, x_s(\phi_2(t))) \leq \delta \ a(t) \\ -\delta^* = \ -\delta \int_0^t \ a(s)ds &\leq y_s(t) - y_s(0) + \int_0^t \ f_2(s, x_s(\phi_2(s)))ds \leq \delta \int_0^t \ a(s)ds = \delta^* \\ -\delta^* &\leq y_s(t) - y_0 + \int_0^t \ f_2(s, x_s(\phi_2(s)))ds \leq \ \delta^*. \end{aligned}$$

Let $max\ \{|x_s(t)-x_0+\int_0^t\ f_1(s,y_s(\phi_1(s)))ds|,\ |y_s(t)-y_0+\int_0^t\ f_2(s,x_s(\phi_2(s)))ds|\}\ <\ \delta.$ Now,

$$\begin{aligned}
|x(t)-x_s(t)| &= \left|x_0+\int_0^t\ f_1(s,y(\phi_1(s)))ds-x_s(t)\right| \\
&\leq \left|x_0+\int_0^t\ f_1(s,y(\phi_1(s)))ds-x_0-\int_0^t\ f_1(s,y_s(\phi_1(s)))ds\right. \\
&+ \left|x_s(t)-x_0+\int_0^t\ f_1(s,y_s(\phi_1(s)))ds\right| \\
&\leq \int_0^t |f_1(s,y(\phi_1(s)))-f_1(s,y_s(\phi_1(s)))|ds+\delta^*,
\end{aligned}$$

then

$$\|x-x_s\|^* \quad \leq \quad b_1^*\ \|y-y_s\|^*+\delta^*.$$

Similarly,

$$|y(t)-y_s(t)| \quad \leq \quad \int_0^t |f_1(s,x(\phi_2(s)))-f_1(s,x_s(\phi_2(s)))|ds+\delta^*,$$

then

$$\|y-y_s\|^* \quad \leq \quad b_2^*\|x-x_s\|^*+\delta^*.$$

Hence

$$\begin{aligned}
\|x-x_s\|^* &\leq b_1^*\ (\ b_2^*\ \|x-x_s\|^*\ +\ \delta^*)\ +\delta^* \\
&\leq \frac{b_1^*\ \delta^*\ \ +\delta^*}{1-b_1^*b_2^*}
\end{aligned}$$

and

$$\|y-y_s\|^* \quad \leq \quad \frac{b_2^*\ \delta^*\ \ +\delta^*}{1-b_1^*b_2^*},$$

then

$$max\ \{\ \|x-x_s\|^*,\ \|y-y_s\|^*\} \quad \leq \quad \frac{b^*\ \delta^*\ \ +\delta^*}{1-b^{*2}}\ =\ \epsilon.$$

Since

$$\begin{aligned}
\|(x,y)-(x_s,y_s)\|_U &= \|(x-x_s),(y-y_s)\|_U \\
&= max\ \{\ \|(x-x_s)\|^*,\ \|(y-y_s)\|^*\ \}<\epsilon,
\end{aligned}$$

then

$$\|(x,y)-(x_s,y_s)\|_U<\epsilon.$$

Example.
Taking into account the equation

$$\frac{dx}{dt}=\frac{t\ e^{-t}}{3}+\frac{(t\ e^{-t}-e^{-t})|y(t)|}{8},\ t\in(0,\infty) \tag{5.2}$$

and

$$\frac{dy}{dt}=\frac{t\ e^{-t}}{4}+\frac{(t\ e^{-t}-e^{-t})|x(t)|}{16},\ t\in(0,\infty). \tag{5.3}$$

Set

$$f_1(t,x) = \frac{t\ e^{-t}}{3} + \frac{(t\ e^{-t} - e^{-t})|y(t)|}{8},$$
$$f_2(t,y) = \frac{t\ e^{-t}}{4} + \frac{(t\ e^{-t} - e^{-t})|x(t)|}{16}.$$

Putting

$$m_1^* = \frac{1}{3},\ m_2^* = \frac{1}{4}, m^* = \ max\{\frac{1}{3},\ \frac{1}{4}\} \ = \ \frac{1}{3}$$
$$b_1^* = \frac{1}{8},\ b_2^* = \frac{1}{16}, b^* = \ max\{\frac{1}{8},\ \frac{1}{16}\} \ = \ \frac{1}{8}$$

we can find that

$$b^* \ = \ 0.125 < 1,$$

then the problem (5.2)-(5.3) has at least one solution $(x,\ y) \in BC(R_+) \times BC(R_+)$.

6. Conclusions

In this investigation, the asymptotic stability and dependency of the solutions for differential equation have been established on R_+. Firstly, we studied the existences of solutions $(x, y) \in BC(R_+) \times BC(R_+)$ of the problem (1.1)-(1.2), by applying the technique associated with the MNC in the Banach space $BC(R_+)$. Next, we studied the asymptotic stability and dependency of the solution $(x, y) \in BC(R_+) \times BC(R_+)$ on the initial data $x_0,\ y_0$ and on the functions $f_i,\ \phi_i$. Moreover, we studied the Hyers-Ulam stability. Finally, we discussed the example to illustrate our results.

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