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HEMI EQUILIBRIUM PROBLEMS ON HADAMARD MANIFOLDS

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Abstract. Throughout the years, equilibrium problems have been used to study various problems appearing in different fields of mechanics, physics, nonlinear programming, engineering mathematics, and so on (see, for example [7], [12]). Consequently, lots of research has been done on solving equilibrium problems under different circumstances in reflexive Banach spaces. This paper contemplates a generalized category of equilibrium problems on Hadamard manifolds called hemiequilibrium problems (HEP). We initiate the existence of solutions to hemiequilibrium problems (HEP) under the monotonicity assumption on the underlying bifunction by applying the KKM technique. We construct some counterexamples in the Hadamard manifold to rationalize our efforts. Additionally, we investigate a few iterative algorithms to solve hemiequilibrium problems on these nonlinear domains. Some particular instances of hemiequilibrium problems are demonstrated. These general classes of equilibrium problems are new on Hadamard manifolds. We hope our outcomes and ideas will spark further investigation in this fascinating and captivating field of research.

1. Introduction

Equilibrium problems have significant applications in many mathematical problems such as optimization problems, variational inequality problems, fixed point problems, Nash equilibria problems, complementarity problems, etc. It also renders us a unified framework to study a wide class of problems arising in economics, finance, network analysis, transportation and optimization theory (see, for example [7], [12]). Numerous findings pertaining to the existence of solutions for equilibrium problems and variational inequality problems have been studied in recent decades (as an illustration, see [2], [3], [16], [32], [23]).
However, a number of academics have recently become interested in applying some concepts and methods of nonlinear analysis from Euclidean spaces to Riemannian

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manifolds. From the point of view of Riemannian geometry, nonconvex and nonsmooth constrained optimization problems can be regarded as convex and smooth unconstrained optimization problems, which offers some advantages for a generalization of optimization techniques from Euclidean spaces to Riemannian manifolds; see, for instance ([30], [26], [24], [4]).

Colao et al. [5] have illustrated an example of an equilibrium problem on an Euclidean space that cannot be solved by using the classical results known in vector spaces, but the problem can be resolved by rewriting it on a Riemannian manifold. On Riemannian manifolds, Németh [18] and Wang et al. [31] investigated monotone and accretive vector fields. From Banach spaces to Hadamard manifolds, Li et al. [13] extended maximal monotone vector fields. Some fundamental existence and uniqueness theorems from the traditional theory of variational inequalities on Euclidean spaces were extended to Hadamard manifolds by Németh [17]. Li et al. [14] introduced the variational inequality problems on Riemannian manifolds. Zhou and Huang [33] provided the notion of KKM mapping and proved a generalized KKM theorem on Hadamard manifolds.

The relationship between a vector variational inequality problem and a vector optimization problem on a Hadamard manifold was established by Zhou and Huang [34]. Tang et al. [28] introduced the proximal point algorithm for pseudomonotone variational inequalities on Hadamard manifolds. Li and Huang [15], studied the generalized vector quasi-equilibrium problems.

Another important and useful generalization of equilibrium problems is known as hemiequilibrium problems (see for example [20], [21], [22]). Hemiequilibrium problems include hemivariational inequality problems, variational inequality problems and equilibrium problems as particular cases. Hemivariational inequality problems have been studied on Hadamard manifolds by Tang et al. [29].

As far as we are aware, there is not a study that addresses hemiequilibrium issues on Hadamard manifolds. The circumstances in which the solution sets of hemiequilibrium problems are nonempty have been determined in this context. Additionally, we have looked at an iterative technique for handling hemiequilibrium problems.

This work serves as an introduction to hemiequilibrium problems in nonlinear spaces, and the insights it presents will motivate researchers to carry out additional research in this intriguing field.

2. Preliminaries

In this segment, we review the essential terminology, fundamental characteristics, and notations required for a thorough understanding of this article. These are included in every textbook on Riemannian geometry (such as [27], [30]).

Let M be an n-dimensional connected manifold. We designate the n-dimensional tangent space of M at x as T_xM, and the tangent bundle of M as $TM = \cup_{x\in M} T_xM$, respectively. M is a Riemannian manifold when it has a Riemannian metric $< .,. >$ on the tangent space T_xM, with a corresponding norm indicated by $\|.\|$. The length of a piecewise smooth curve $\gamma : [a, b] \to M$ joining x to y such that $\gamma(a) = x$ and $\gamma(b) = y$, is defined by $L(\gamma) = \int_a^b \| \dot{\gamma}(t) \|_{\gamma(t)} \, dt$. Then for any $x, y \in M$ the original topology on M is induced by the Riemannian distance $d(x, y)$, which can be defined as the minimum length of all curves connecting x and y.

For any vector fields X, Y on M, there is precisely one covariant derivation known

as the Levi-Civita connection, denoted by $\nabla_X Y$ on any Riemannian manifold. Consider a smooth curve in M called γ. If $\nabla_{\gamma'} X = 0$, then a vector field X is said to be parallel along γ. We refer to γ as a geodesic if γ' is parallel along γ. If the length of a geodesic connecting x and y in M equals $d(x, y)$, then it is considered as a minimal geodesic.
A Riemannian manifold is complete if for any $x \in M$ all geodesics emanating from x are defined for all $t \in \mathbb{R}$. By the Hopf-Rinow theorem, we know that if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover, (M, d) is a complete metric space and bounded closed subsets are compact.
Assuming that M is complete the exponential mapping $\exp_x : T_x M \to M$ is defined by $\exp_x v = \gamma_v(1)$, where γ_v is the geodesic defined by its position x and velocity v at x.
Recall that a Hadamard manifold is a simply connected complete Riemannian manifold with nonpositive sectional curvature. On the Hadamard manifold, the exponential mapping exp and its inverse $\exp^{-1}$ are continuous.

2.1. **Convexity.** Let M represent a Hadamard manifold with finite dimensions.

Definition 2.1. *([26]) A subset K of M is said to be geodesic convex if and only if for any two points $x, y \in K$, the geodesic joining x to y is contained in K. That is if $\gamma : [0, 1] \to M$ is a geodesic with $x = \gamma(0)$ and $y = \gamma(1)$, then $\gamma(t) \in K,\ for\ 0 \leq t \leq 1$.*

Definition 2.2. *([26]) A real-valued function $f : M \to \mathbb{R}$ defined on a geodesic convex set K is said to be geodesic convex if and only if for $0 \leq t \leq 1$,*

$$f(\gamma(t)) \leq (1 - t)f(\gamma(0)) + tf(\gamma(1)).$$

Definition 2.3. *([5]) For an arbitrary subset $C \subseteq M$ the minimal geodesic convex subset which contains C is called the convex hull of C and is denoted by $co(C)$. It is easy to check that $co(C) = \bigcup_{n=1}^{\infty} C_n$, where $C_0 = C$ and $C_n = \{z \in \gamma_{x,y} : x, y \in C_{n-1}\}$.*

2.2. **Locally Lipschitz function.** Let M represent a finite dimensional Hadamard manifold.

Definition 2.4. *([1], ([10]), ([25])) Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a proper function. It is said to be a locally Lipschitz function on M if for each $x \in domf$, there exist ϵ_x and $L_x > 0$ such that*

$$|f(z) - f(y)| \leq L_x d(z, y),\ \forall z, y \in B(x, \epsilon_x),$$

where denotes an open ball centered in $x \in M$ and radius ϵ_x.

Definition 2.5. *([10]) Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a locally Lipschitz function on M. Given $x \in domf$, the generalized directional derivative in the sense Clarke of f at the point p in the direction $w \in T_p M$, denoted by $f^o(p; w)$, is defined as*

$$f^o(p; w) = \lim_{t \to 0^+} \sup_{q \to p} \frac{fo\phi^{-1}(\phi(q) + td\phi(p)w) - fo\phi^{-1}(\phi(q))}{t}$$

where (ϕ, U) is a chart at p.

We require the following lemma which provides some fundamental characteristics of the generalized directional derivative on Hadamard manifolds.

Lemma 2.1. *([10]) Let M be a Riemannian manifold and $p \in M$. Suppose that the function $f : M \to R$ is Lipschitz of rank K on an open neighborhood U of p. Then,*

(i) *for each $q \in U$, the function $w \to f^o(q; w)$ is finite, positive homogeneous and subadditive on T_qM, and satisfies*

$$|f^o(q; w)| \leq K\|w\|;$$

(ii) *$f^o(q; w)$ is upper semicontinuous on TM and as a function of w alone is Lipschitz of rank K on T_qM for each $q \in U$;*

(iii) *$f^o(q; -w) = (-f)^o(q; w)$ for each $q \in U$ and $w \in T_qM$.*

Unless otherwise indicated, we consider M to be a finite dimensional Hadamard manifold and $K \subseteq M$ to represent a nonempty closed geodesic convex set in the remainder portion of the work.

3. Existence Results for Hemiequilibrium Problems

This section deals with the existence of solutions to hemiequilibrium problems (HEP) on Hadamard manifolds.

Let M be a Hadamard manifold and K be a closed geodesic convex subset of M. Let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying the property $F(u, u) = 0$ for all $u \in K$. Then the equilibrium problem introduced by Colao et al. [5] is to find a point $u \in K$, such that

$$(EP)\ F(u, v) \geq 0\ for\ all\ v \in K. \tag{1}$$

We present hemiequilibrium problem on Hadamard manifolds.

Assume that M is a Hadamard manifold and K is a closed geodesic convex subset of M. Let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying the property $F(u, u) = 0$ for all $u \in K$. Let $J : M \to \mathbb{R}$ be a locally Lipschitz function. Then the hemiequilibrium problem, denoted by HEP(F,J,K) is to find an element $u \in K$ such that

$$F(u, v) + J^o(u; \exp_u^{-1} v) \geq 0,\ \forall v \in K. \tag{2}$$

Definition 3.6. *([5]) We call a bifunction F to be monotone on K, if for any $u, v \in K$, we have*

$$F(u, v) + F(v, u) \leq 0. \tag{3}$$

Definition 3.7. *Let K be a geodesic convex subset of M. A function $f : K \to \mathbb{R}$ is said to be hemicontinuous if for every geodesic $\gamma : [0, 1] \to K$, whenever $t \to 0$, $f(\gamma(t)) \to f(\gamma(0))$.*

To demonstrate the existence theorem, we revisit the idea of the KKM lemma.

Definition 3.8. *([34]) Let $K \subset M$ be a nonempty closed geodesic convex set and $G : K \to 2^K$ be a set-valued mapping. We say that G is a KKM mapping if for any $\{u_1, ..., u_m\} \subset K$, we have*

$$co(\{u_1, ..., u_m\}) \subset \bigcup_{i=1}^{m} G(u_i).$$

Lemma 3.2. *([5]) Let K be a nonempty closed geodesic convex set and $G : K \to 2^K$ be a set-valued mapping such that for each $u \in K$, $G(u)$ is closed. Suppose that*

(i) *there exists $u_0 \in K$ such that $G(u_0)$ is compact.*
(ii) *$\forall u_1, .., u_m \in K$, $co(\{u_1, .., u_m\}) \subset \bigcup_{i=1}^{m} G(u_i)$.*

Then $\bigcap_{u\in K} G(u) \neq \emptyset$.

We then provide the lemma that follows, which is required for the sequel.

Lemma 3.3. *Let* $F : K \times K \to \mathbb{R}$ *be monotone and hemicontinuous in the first argument. Let for fixed* $u \in K$*, the mapping* $z \mapsto F(u, z)$ *be geodesic convex and* $J : M \to \mathbb{R}$ *be a locally Lipschitz function. Then* $u \in K$ *satisfies*

$$F(u, v) + J^o(u; \exp_u^{-1} v) \geq 0, \ \forall v \in K; \tag{4}$$

if and only if

$$-F(v, u) + J^o(u; \exp_u^{-1} v) \geq 0, \ \forall v \in K. \tag{5}$$

Proof. Since F is monotone

$$F(u, v) \leq -F(v, u).$$

Therefore (4) implies (5). Conversely, let $u \in K$ be a solution of (5). Then

$$-F(v, u) + J^o(u; \exp_u^{-1} v) \geq 0, \ \forall v \in K. \tag{6}$$

Let $w \in K$ be arbitrarily fixed and $\gamma(t) = \exp_u(t \exp_u^{-1} w)$ for $t \in [0, 1]$ be a geodesic joining u and w.
As K is geodesic convex, then $\gamma(t) \in K$, for $t \in [0, 1]$. It follows from (6)

$$-F(\gamma(t), u) + J^o(u; \exp_u^{-1} \gamma(t)) \geq 0 \ for \ 0 \leq t \leq 1. \tag{7}$$

Now

$$0 = F(\gamma(t), \gamma(t)) \leq tF(\gamma(t), w) + (1-t)F(\gamma(t), u), \ (as\ z \mapsto F(u, z)\ is\ geodesic\ convex,)$$

$$\Rightarrow t[F(\gamma(t), u) - F(\gamma(t), w)] \leq F(\gamma(t), u)),$$

$$\Rightarrow t[F(\gamma(t), u) - F(\gamma(t), w)] \leq J^o(u; \exp_u^{-1} \gamma(t)), \ by\ (7);$$

$$\Rightarrow t[F(\gamma(t), u) - F(\gamma(t), w)] - J^o(u; t \exp_u^{-1} w) \leq 0;$$

by the positively homogeneous property of $J^o(u; t \exp_u^{-1} w)$ [see Lemma 2.1], we have

$$\Rightarrow t[F(\gamma(t), u) - F(\gamma(t), w)] - tJ^o(u; \exp_u^{-1} w) \leq 0,$$

$$\Rightarrow F(\gamma(t), u) - F(\gamma(t), w) - J^o(u; \exp_u^{-1} w) \leq 0, \ (as\ t > 0).$$

Since F is hemicontinuous in the first argument taking $t \to 0$, we have

$$F(u, u) - F(u, w) - J^o(u; \exp_u^{-1} w) \leq 0, \ for\ all\ w \in K.$$

$$\Rightarrow F(u, w) + J^o(u; \exp_u^{-1} w) \geq 0, \ for\ all\ w \in K.$$

This completes the proof. □

Now, the primary existence theorem needs to be proven. First, we take the set K to be bounded. In this case, K is a compact subset.

Theorem 3.1. *Let* K *be a compact subset of* M *and* $F : K \times K \to \mathbb{R}$ *be monotone and hemicontinuous in the first argument. Suppose for fixed* $u \in K$*, the mapping* $z \mapsto F(u, z)$ *is geodesic convex, lower semicontinuous and* $J : M \to \mathbb{R}$ *be a locally Lipschitz function. Then HEP(F,J,K) admits a solution.*

Proof. Consider the two set-valued mappings $G_1 : K \to 2^K$ and $G_2 : K \to 2^K$ such that

$$G_1(v) = \{u \in K : F(u,v) + J^o(u; \exp_u^{-1} v) \geq 0\},\ for\ all\ v \in K,$$

$$G_2(v) = \{u \in K : -F(v,u) + J^o(u; \exp_u^{-1} v) \geq 0\},\ for\ all\ v \in K.$$

It is easy to see that $u \in K$ solves HEP(F,J,K) if and only if $u \in \cap_{v\in K} G_1(v)$. Thus it suffices to prove that $\cap_{v\in K} G_1(v) \neq \emptyset$.

Step-1: G_1 is a KKM map.

So we have to prove that for any choice of $v_1, v_2, ..., v_m \in K$,

$$co(\{v_1, ..., v_m\}) \subset \bigcup_{i=1}^{m} G_1(v_i). \tag{8}$$

Suppose on the contrary that there exists a point $u_0\ in\ K$, such that $u_0 \in co(\{v_1, ..., v_m\})$ but $u_0 \notin \bigcup_{i=1}^m G_1(v_i)$. That is

$$F(u_0, v_i) + J^o(u_0; \exp_{u_0}^{-1} v_i) < 0,\ \forall i \in \{1, ..., m\}. \tag{9}$$

This implies that for any $i \in \{1, ..., m\}$, $v_i \in \{v \in K : F(u_0, v) + J^o(u_0; \exp_{u_0}^{-1} v) < 0\}$. Since the function $v \mapsto F(u_0, v)$ is geodesic convex, the set $\{v \in K : F(u_0, v) + J^o(u_0; \exp_{u_0}^{-1} v) < 0\}$ is a geodesic convex set. Then

$$u_0 \in co(\{v_1, ..., v_m\}) \subseteq \{v \in K : F(u_0, v) + J^o(u_0; \exp_{u_0}^{-1} v) < 0\}.$$

Therefore $F(u_0, u_0) + J^o(u_0; \exp_{u_0}^{-1} u_0) < 0$.

But we have $F(u_0, u_0) + J^o(u_0; \exp_{u_0}^{-1} u_0) = 0$, a contradiction. Hence G_1 is a KKM mapping.

Step:2 G_2 is a KKM map.

From Lemma 3.3, we have $G_1(v) \subset G_2(v),\ \forall v \in K$. That is,

$$co(\{v_1, v_2, ..., v_m\}) \subset \bigcup_{i=1}^{m} G_2(v_i).$$

Hence G_2 is also a KKM mapping.

Step:3 $G_2(v)$ is closed.

Let $\{u_n\} \in G_2(v)$, such that $u_n \to u$ as $n \to \infty$. We show that $u \in G_2(v)$, Since $\{u_n\} \in G_2(v)$, we have

$$-F(v, u_n) + J^o(u_n; \exp_{u_n}^{-1} v) \geq 0.$$

Since $F(v, .)$ is lower semicontinuous and J^o is Lipschitz continuous, we have

$$-F(v, u) + J^o(u; \exp_u^{-1} v) \geq 0.$$

Hence $u \in G_2(v)$. That is $G_2(v)$ is closed for all $v \in K$.

Step:4 $G_2(v)$ is compact.

Since $G_2(v)$ is a closed subset of a compact set K. So $G_2(v)$ is compact for all $v \in K$.

Hence by Lemma 3.2, there exists a point $u \in K$ such that $u \in \bigcap_{v\in K} G_2(v)$.

By Lemma 3.3, we have $\bigcap_{v\in K} G_1(v) = \bigcap_{v\in K} G_2(v)$. That is $u \in \bigcap_{v\in K} G_1(v)$.

So there exists a point $u \in K$ such that

$$F(u, v) + J^o(u; \exp_u^{-1} v) \geq 0,\ \forall v \in K.$$

Therefore, $u \in K$ solves HEP(F,K,J). □

Example 3.1. *Let K be a subset of $\mathbb{R}^2$ defined by*

$$K = \{u = (u_1, u_2) \in \mathbb{R}^2_+ : u_1^2 + u_2^2 \le 4 \le (u_1 - 1)^2 + u_2^2\}.$$

K is not convex in $\mathbb{R}^2$ (see Example 3.1 of [29]*). We consider the Poincare upper half model*

$$H^2 = \{u = (u_1, u_2) \in \mathbb{R}^2; u_2 > 0\};$$

which forms a Hadamard manifold with constant curvature -1. Now the set $K \subset H^2$ is geodesic convex and compact with respect to the metric defined by $g_{H^2} = \frac{\delta_{ij}}{u_2^2}$. Now we define the bifunction $F : K \times K \to \mathbb{R}$ by

$$F(u, v) = v_2 - u_2.$$

Now $F(u, v) + F(v, u) = v_2 - u_2 + u_2 - v_2$;
Hence F is monotone on K.
It is clear that $z \mapsto F(u, z)$ is geodesic convex, lower semicontinuous.
Let $J : H^2 \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$J(u) = \ln u_2.$$

Then J is locally Lipschitz on H^2 (see Example 5 of [8]*).*
Then by Theorem 3.1, the HEP(F,J,K) has a solution.

The situation when K is unbounded, or noncompact, is the next scenario we consider.

Theorem 3.2. *Let K be a noncompact subset of M and $F : K \times K \to \mathbb{R}$ be monotone and hemicontinuous in the first argument. Suppose for fixed $u \in K$, the mapping $z \mapsto F(u, z)$ is geodesic convex, lower semicontinuous and $J : M \to \mathbb{R}$ be a locally Lipschitz function. If there exists a point $v_0 \in K$, such that*

$$F(u, v_0) + J^o(u; \exp_u^{-1} v_0) < 0, \text{ whenever } d(\boldsymbol{0}, u) \to +\infty, \ u \in K, \tag{10}$$

then HEP(F,J,K) has a solution.

Proof. Given a point $\mathbf{0} \in M$, we denote $\Sigma_R = \{u \in M : d(\mathbf{0}, u) \le R\}$ to be the closed geodesic ball of radius R and center $\mathbf{0}$. Let $K_R = K \cap \Sigma_R$. If $K_R \neq \emptyset$, then there exists at least one $u_R \in K_R$ such that

$$F(u_R, v) + J^o(u_R; \exp_{u_R}^{-1} v) \ge 0, \ \forall v \in K_R, \tag{11}$$

by Theorem 3.1.
We now take a point $v_0 \in K$ satisfying (10), with $d(\mathbf{0}, v_0) < R$, so $v_0 \in K_R$.
Hence by (11), we have

$$F(u_R, v_0) + J^o(u_R; \exp_{u_R}^{-1} v_0) \ge 0. \tag{12}$$

If $d(\mathbf{0}, v_R) = R$ for all R, we may choose R large enough so that $d(\mathbf{0}, v_R) \to +\infty$.
Hence by (10), $F(u_R, v_0) + J^o(u_R; \exp_{u_R}^{-1} v_0) < 0$, contradicts (12).
So there exists an R such that $d(\mathbf{0}, v_R) < R$.
Given $v \in K$, let $\gamma(t) = \exp_{u_R}(t \exp_{u_R}^{-1} v)$ be a geodesic joining u_R to v. Now since $d(\mathbf{0}, u_R) < R$, we can choose $0 < t < 1$, sufficiently small so that $\gamma(t) \in K_R$.
Hence $0 \le F(u_R, \gamma(t)) + J^o(u_R; \exp_{u_R}^{-1} \gamma(t))$
$\le tF(u_R, v) + (1 - t)F(u_R, u_R) + J^o(u_R; t \exp_{u_R}^{-1} v)$
$= t[F(u_R, v) + J^o(u_R; \exp_{u_R}^{-1} v)], \ [by \ Lemma \ 2.1]$
or, as $t > 0$ $F(u_R, v) + J^o(u_R; \exp_{u_R}^{-1} v) \ge 0, \ for \ v \in K$.
That is u_R solves HEP(F,J,K). □

4. Proximal Point Algorithm for Solving Hemiequilibrium Problem

Several iterative techniques for resolving hemiequilibrium problems on linear spaces were proposed by Noor et al. [22]. Neto et al. [6] have analyzed the proximal point algorithm for optimization problems involving monotone vector fields on Hadamard manifolds. They have given examples of many nonconvex and nonmonotone functions which can be transformed into convex and monotone functions respectively with the help of proper matrices. For equilibrium problems on Hadamard manifolds, Noor et al. [19] haveoffered an implicit iterative (proximal point) approach. On these spaces, Jana and Nahak [11] have investigated a few techniques for solving mixed equilibrium issues. The proximal point algorithm (PPA) for the hemiequilibrium problem (2) will now be addressed.
We assume K to be compact geodesic convex subset of the Hadamard manifold M in this section.
At stage n, given $u_n \in K$, $\rho > 0$, compute $u_{n+1} \in K$, as a solution of the hemiequilibrium problem

$$F(u_{n+1}, v) + \frac{1}{\rho}\left\langle \exp_{u_n}^{-1} u_{n+1}, \exp_{u_{n+1}}^{-1} v \right\rangle + J^o(u_{n+1}; \exp_{u_{n+1}}^{-1} v) \geq 0, \ \forall v \in K. \quad (13)$$

Definition 4.9. *([22]) $J^o(;)$ is said to be monotone if*

$$J^o(u; \exp_u^{-1} v) + J^o(v; \exp_v^{-1} u) \leq 0. \quad (14)$$

Let us recall that a geodesic triangle $\Delta(x_1 x_2 x_3)$ of a Riemannian manifold is the set consisting of three distinct points x_1, x_2, x_3 called the vertices and three minimizing geodesic segments γ_{i+1} joining x_{i+1} to x_{i+2} called the sides, where $i = 1, 2, 3 (mod\ 3)$.

Theorem 4.3. [27] *Let M be a Hadamard manifold, $\Delta(x_1 x_2 x_3)$ a geodesic triangle and $\gamma_{i+1} : [0, l_{i+1}] \to M$ geodesic segments joining x_{i+1} to x_{i+2} and set $l_{i+1} = l(\gamma_{i+1})$, $\theta_{i+1} = \measuredangle(\gamma'_{i+1}(0), -\gamma'_i(l_i))$, for $i = 1, 2, 3 (mod\ 3)$. Then*

$$\theta_1 + \theta_2 + \theta_3 \leq \pi,$$

$$l_{i+1}^2 + l_{i+2}^2 - 2 l_{i+1} l_{i+2} \cos\theta_{i+2} \leq l_i^2,$$

$$d^2(x_{i+1}, x_{i+2}) + d^2(x_{i+2}, x_i) - 2\left\langle \exp_{x_{i+2}}^{-1} x_{i+1}, \exp_{x_{i+2}}^{-1} x_i \right\rangle \leq d^2(x_i, x_{i+1}). \quad (15)$$

By using the above inequality for any three points $x, y, z \in M$, we can get

$$d^2(x, y) \leq \left\langle \exp_x^{-1} z, \exp_x^{-1} y \right\rangle + \left\langle \exp_y^{-1} z, \exp_y^{-1} x \right\rangle. \quad (16)$$

Lemma 4.4. *([13]) Let $x_0 \in M$ and $\{x_n\} \in M$ such that $x_n \to x_0$. Then the following assertions hold.*

(i) *For any $y \in M$*

$$\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y \ \text{and} \ \exp_y^{-1} x_n \to \exp_y^{-1} x_0.$$

(ii) *If $\{v_n\}$ is a sequence such that $v_n \in T_{x_n}M$ and $v_n \to v_0$, then $v_0 \in T_{x_0}M$.*

(iii) *Given the sequence $\{u_n\}$ and $\{v_n\}$ with $u_n, v_n \in T_{x_n}M$, if $u_n \to u_0$ and $v_n \to v_0$ with $u_0, v_0 \in T_{x_0}M$, then $\langle u_n, v_n \rangle \to \langle u_0, v_0 \rangle$.*

The principle of Fejér convergence and the associated findings, which are available in [9] and [13], are then revisited.

Definition 4.10. *Let X be a complete metric space and $A \subseteq X$ be a nonempty set. A sequence $\{x_n\} \subset X$ is said to be Fejér convergent to A if*

$$d(x_{n+1}, y) \leq d(x_n, y), \ \forall y \in A \text{ and } n = 0, 1, 2, ...$$

Lemma 4.5. *Let X be a complete metric space and let A be a nonempty subset of X. Suppose $\{x_n\} \subset X$ be Fejér convergent to K and any cluster point of $\{x_n\}$ belongs to A. Then $\{x_n\}$ converges to a point of A.*

We are now in a position to prove the convergence of PPA for hemiequilibrium problems involving monotone vector fields.

Theorem 4.4. *Let F be monotone and continuous in the first argument and SOL(HEP) $\neq \emptyset$. Also assume that the sequence $\{u_n\}$ generated by (13) is well defined and $J^o(;)$ is monotone. Then $\{u_n\}$ converges to a solution of the hemiequilibrium problem (2).*

Proof. We first proof that $\{u_n\}$ is Fejér convergent to SOL(HEP). Let $v \in K$ be a solution of (2). Then

$$F(u, v) + J^o(u; \exp_u^{-1} v) \geq 0, \forall v \in K. \tag{17}$$

Taking $v = u_{n+1}$ in (17), we get

$$F(u, u_{n+1}) + J^o(u; \exp_u^{-1} u_{n+1}) \geq 0. \tag{18}$$

Since F is monotone then

$$F(u_{n+1}, u) \leq -F(u, u_{n+1}). \tag{19}$$

From (13), taking $v = u$ we have

$$F(u_{n+1}, u) + \frac{1}{\rho}\left\langle \exp_{u_n}^{-1} u_{n+1}, \exp_{u_{n+1}}^{-1} u \right\rangle + J^o(u_{n+1}; \exp_{u_{n+1}}^{-1} u) \geq 0;$$

$$\Rightarrow \frac{1}{\rho}\left\langle \exp_{u_n}^{-1} u_{n+1}, \exp_{u_{n+1}}^{-1} u \right\rangle \geq -[F(u_{n+1}, u) + J^o(u_{n+1}; \exp_{u_{n+1}}^{-1} u)];$$

$$\Rightarrow \frac{1}{\rho}\left\langle \exp_{u_{n+1}}^{-1} u_n, \exp_{u_{n+1}}^{-1} u \right\rangle \leq F(u_{n+1}, u) + J^o(u_{n+1}; \exp_{u_{n+1}}^{-1} u),$$

$\leq -F(u, u_{n+1}) - J^o(u; \exp_u^{-1} u_{n+1})$ *(by monotonicity)*
≤ 0.
So we finally get as $\rho > 0$,

$$\left\langle \exp_{u_{n+1}}^{-1} u_n, \exp_{u_{n+1}}^{-1} x \right\rangle \leq 0. \tag{20}$$

Considering the geodesic triangle $\Delta(u_n u_{n+1} u)$ from (15), we get

$$d^2(u_{n+1}, u) + d^2(u_{n+1}, u_n) - 2\left\langle \exp_{u_{n+1}}^{-1} u_n, \exp_{u_{n+1}}^{-1} u \right\rangle \leq d^2(u_n, u).$$

It follows from (20)

$$d^2(u_{n+1}, u) + d^2(u_{n+1}, u_n) \leq d^2(u_n, u). \tag{21}$$

This clearly implies that $d^2(u_{n+1}, u) \leq d^2(u_n, u)$, so $\{u_n\}$ is Fejér convergent to SOL(HEP). From (21) it follows that

$$d^2(u_{n+1}, u_n) \leq d^2(u_n, u) - d^2(u_{n+1}, u). \tag{22}$$

Since the sequence $\{d(u_n, u)\}$ is bounded and monotone, it is also convergent. Hence by (22), $\lim_{n\to\infty} d^2(u_{n+1}, u_n) = 0$. That is

$$\lim_{n\to\infty} d(u_{n+1}, u_n) = 0.$$

Next we prove that any cluster point of $\{u_n\}$ belongs to SOL(HEP). Let u be a cluster point of $\{u_n\}$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $u_{n_k} \to u$. Hence $d(u_{n_k+1}, u_{n_k}) \to 0$, by the assertion just proved, and so $u_{n_k+1} \to u$. It follows from (13) with $n = n_k$,

$$F(u_{n_k+1}, v) + \frac{1}{\rho}\big\langle \exp^{-1}_{u_{n_k}} u_{n_k+1}, \exp^{-1}_{u_{n_k+1}} v\big\rangle + J^o(u_{n_k+1}; \exp^{-1}_{u_{n_k+1}} v) \geq 0, \forall v \in K. \tag{23}$$

Passing to the limit as $k \to \infty$ in (23) we get

$$F(u, v) + J^o(u_{n+1}; \exp^{-1}_{u_{n+1}} v) \geq 0, \forall v \in K.$$

That is $u \in SOL(HEP)$. Hence by Lemma 4.5, $\{u_n\}$ converges to point of SOL(HEP). This completes the proof. □

5. Applications

We will next go over a few specific instances of hemiequilibrium problems.

(i) Hemivariational inequality problem: For each $u \in K$, let $A : K \to TM$ be a vector field, that is, $A(u) \in T_uM$. Tang et al. ([29]), proposed hemivariational inequality problem on K, which is to find a point $u \in K$ such that

$$\big\langle A(u), \exp^{-1}_u v\big\rangle + J^o(u; \exp^{-1}_u v) \geq 0, \ \forall v \in K. \tag{24}$$

If we define

$$F(x, y) = \big\langle A(u), \exp^{-1}_u v\big\rangle,$$

then the hemiequilibrium problem (2) and the hemivariational inequality problem (24) are equivalent.

(ii) Equilibrium problem: If the function J is constant, then $J^o(u; .) = 0 \in T_uM$. So HEP(F,J,K) reduces to the following equilibrium problem introduced by Colao et al. ([5]), which is to find $u \in K$ such that

$$F(u, v) \geq 0, \ for\ all\ v \in K,$$

(iii) Variational inequality problem: Assume that $V : K \to TM$ is a vector field, that is, $V_u \in T_uM$ for each $u \in K$. If the function J is constant, then $J^o(u; .) = 0 \in T_uM$.
Then the problem introduced by Németh ([17]), is to find $u \in K$ such that

$$\big\langle V_u, \exp^{-1}_u v\big\rangle \geq 0, \ \forall v \in K, \tag{25}$$

is called a variational inequality problem on K. If we denote

$$F(x, y) = \big\langle V_u, \exp^{-1}_u v\big\rangle,$$

then the hemiequilibrium problem (2) and the variational inequality problem (25) are same.

(iv) If M is a linear space, then HEP(F,J,K) is to find a point $u \in K$ such that

$$F(u, v) + J^o(u; v - u) \geq 0, \ \forall v \in K,$$

which is a hemiequilibrium problem on Banach spaces introduced and investigated by Noor ([20], [21], [22]).

(v) Optimization problem: Let $f : K \to \mathbb{R}$ be a function and consider the minimization problem

$$(P)\ find\ x \in K\ such\ that\ f(x) = \min_{y \in K} f(y).$$

If we set $F(x,y) = f(y) - f(x)$, for all $x, y \in K$. Thus the problems (P) and (15) are identical.

Conclusion: To best of our insight this paper is the first paper to address hemiequilibrium problems on Hadamard manifolds. We acknowledge that there are numerous extension on studying hemiequilibrium problems on nonlinear spaces, for emample

(i) From a theoretical perspective, one can investigate existence results by applying weaker monotonicity assumptions on the underlying bifunctions.
(ii) Different algorithms can be searched for solving hemiequilibrium problems.
(iii) We have demonstrated the results on Hadamard manifolds. One can try to extend these findings on Riemannian manifolds.

This paper can be utilized as a stepping stone to analyze hemiequilibrium problems on Hadamard Manifolds. We anticipate further exploration in this area in near future.

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