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SOME NEW QUALITATIVE RESULTS FOR TWO DIMENSIONAL NONLINEAR DIFFERENTIAL SYSTEMS

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ABSTRACT. As we know ordinary differential equations, systems of ordinary differential equations, in particular, two dimensional nonlinear differential systems have significant and various applications in qualitative theory of ordinary differential equations. In some real world applications, it is needed to have information in relation to the qualitative concepts called stability, boundedness, convergence, etc. of solutions of that kind of mathematical models. Most of time, exact solutions of that kind of equations cannot be obtained explicitly, except numerically. In the pertinent literature, one of the famous method is known the Lyapunov's second method, which allows to have information about qualitative behaviors of solutions without solving the equation understudy. In this study, we deal with a nonlinear a two dimensional nonlinear differential system. We examine uniform asymptotic stability, boundedness, uniform boundedness and uniform-ultimate boundedness of solutions of that two dimensional nonlinear differential system. We will prove three new theorems on the mentioned qualitative concepts by using the Lyapunov's second method. We provide two examples to demonstrate how the results of the study can be applied. The results of this study generalize some recent results, which can be found in the present literature.

1. Introduction

As we know from the relevant literature, ordinary differential equations of second order without and with delay have numerous applications in sciences and engineering, see, for example the books of Ahmad and Rama Mohana Rao [8], Burton [10], Chau [11], Hsu [19], LaSalle and Lefschetz [23], Nakanishi and Seto [25], Yoshizawa ([39, 40]) and the references of these books for some applications of that kind of differential equations. Studying the qualitative characteristics of these kinds of

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mathematical models is therefore merited. Before giving the qualitative results of this study, we would like to outline some works regarding qualitative behaviors of ordinary differential equations of second order.

Ezeilo [12] considered the second-order vector system as follows:

$$X'' + CX' + G(X) = P(t, X, X').$$

Ezeilo [12] obtained sufficient conditions for convergence and ultimate boundedness of solutions of this system by means of the second method of Lyapunov.

Ezeilo [13] considered the following two dimensional nonlinear differential system:

$$X' = F(X) + BY,$$

$$Y' = G(X) + DY.$$

Ezeilo [13] construcded sufficient conditions for the asymptotic stability solutions of this system by using the Lyapunov's second method.

Tejumola [31] studied the scalar nonlinear differential equations of second order as follows:

$$x'' + f(x, x')x' + g(x) = p(t, x, x').$$

Tejumola [31] proved that solutions of this differential equation are all ultimately bounded with the bounding constant dependening only on the functions of this equation.

Qian [29] dealt with the scalar nonlinear differential equation of second order:

$$x'' + (f(x') + k(x)x')\, x' + g(x) = 0.$$

Qian [29] obtained sufficient conditions under which the trivial solution of this equation is globally asymptotic stable by means of the second method of Lyapunov.

Tunç [33] focused on the nonlinear vector differential equation of second order:

$$X'' + B(t)G(X, X')X' + A(t)F(X) = P(t, X, X').$$

Tunç [33] investigated the stability and boundedness of solutions of this vector differential equation of second order by using the second method of Lyapunov, when $P(.) = 0$ and $P(.) \neq 0$, respectively.

Tunç and Dinç [36] studied the boundedness and square integrability of solutions of non-linear systems of differential equations of second order as follows , respectively:

$$(q(t)X')' + H(t, X, X')X' + a(t)X = Q(t, X, X')$$

and

$$(q(t)X')' + \Phi(t, X, X') + a(t)G(X) = Q(t, X, X').$$

The authors [36] established two new theorems, which have sufficient conditions guaranteeing the boundedness and square integrability of solutions these systems. The proofs of the results depend upon the integral test.

Adeyanju and Adams [3] provided certain sufficient conditions that guarantee the stability of zero solution and boundedness of all solutions of the following vector differential equation of second order, when $P(.) = 0$ and $P(.) \neq 0$, respectively:

$$X'' + AX' + H(X) = P(t, X, X').$$

The basic tool in the proofs of the results of Adeyanju and Adams [3] was a suitable Lyapunov function.

Adeyanju [2] considered the following second order nonlinear vector differential equation:

$$X'' + F(X, X')X' + H(X) = P(t, X, X').$$

Adeyanju [2] derived sufficient conditions for the stability and boundedness of solutions of this vector differential equation by using the second method of Lyapunov, when $P(.) = 0$ and $P(.) \neq 0$, respectively.

Adams et al. [1] obtained some criteria for the stability and boundedness of solutions to the following nonlinear scalar differential equation of second order as follows:

$$x'' + b(t)f(x, x') + c(t)g(x)h(x') = p(t, x, x').$$

Adams et al. [1] obtained the results of this study by applying a suitable Lyapunov function.

In addition to the studies already mentioned, numerous intriguing findings can be seen in the articles of ([4, 5, 7, 9, 14, 15, 16, 17, 18, 20, 21, 22, 24, 26, 27, 28, 30, 32, 34, 35, 37, 38]), where stability, exponential stability, asymptotic stability, stability in the large, boundedness, uniform-ultimate boundedness, convergence, integrability, etc. of various mathematical models as ordinary differential equations of second order, delay ordinary differential equations of second order, ordinary differential system of second order, etc. have been investigated in general by the second method of Lyapunov, integral test, and some others. For the sake for the sake of the brevity, we would not like to give more details.

As for the motivation of this study, in 2023, Adeyanju et al. [6] focused on the following systems of first order Aizermann differential equations:

$$\begin{cases} \dot{X} = F(X) + H(Y) + P_1(t, X, Y), \\ \dot{Y} = CX + DY + P_2(t, X, Y), \end{cases} \tag{1}$$

where $X,\ Y \in R^n,\ C,\ D \in R^{n\times n}$ are symmetric constant matrices, $F,\ G \in C^1(R^n, R^n)$ with $F(0) = G(0) = 0$ and $P_1,\ P_2 \in C^1(R^+ \times R^n \times R^n, R^n)$. Adeyanju et al. [6] discussed the uniform asymptotic stability of the trivial solution and uniform ultimate boundedness of all solutions to Aizermann vector differential equation (1) by defining an appropriate complete Lyapunov function. By this work, Adeyanju et al. [6] solved some the open problems contained in Ezeilo [13].

In this study, inspired by the results of Adeyanju et al. [6] and those have been presented above, we deal with the following two dimensional nonautonomus and nonlinear differential system:

$$\begin{cases} \dot{X} = A(t)F(X) + B(t)Y + P_1(t, X, Y), \\ \dot{Y} = C(t)G(X) + D(t)Y + P_2(t, X, Y), \end{cases} \tag{2}$$

where $X, Y \in R^n,\ A, B, C, D \in C(R^+, R^{n\times n})$ are symmetric matrices functions, $F,\ G \in C^1(R^n, R^n)$ with $F(0) = H(0) = 0$ and $P_1,\ P_2 \in C^1(R^+ \times R^n \times R^n, R^n)$. We will study the some qualitative concepts as in Adeyanju et al. [6]. The aim of this is study to extend the results of Adeyanju et al. [3] and allow new contributions to the findings of Adeyanju et al. [6], Ezeilo [13], and those have already been mentioned above.

2. Qualitative results

Let $P_1(.) = P_1(t, X, Y) = 0$ and $P_2(.) = P_2(t, X, Y) = 0$.

Three new theorems will be given here as our new qualitative findings, Theorems 2.1-2.3, respectively.

Theorem 2.1. *Let* $K_*, \alpha, \alpha_i, (i = 0, 1, 2), \delta_k, (k = 0, 1, 2, 3), \pi_j, (j = 1, 2, 3, 4), \gamma_*, \gamma_i, (i = 1, 2, 3, 4), \beta, \beta_j, (j = 0, 1, 2), \Delta_*, \Delta_k, (k = 0, 1, 2, 3)$, *and* μ_1, μ_2 *be some positive constants such that the following conditions hold:*

(i)
$$\delta_0 \le \lambda_i(D(t)A(t)J_f(X) - B(t)C(t)J_g(X) + C(t)J_g(X)) \le \Delta_0,$$
$$\delta_2 \le \left|\lambda_i(B(t)(I - B(t)) + \frac{1}{2}B'(t))\right| \le \Delta_2;$$

(ii)
$$\delta_1 \le \lambda_i(-B(t)) \le \Delta_1,$$
$$\gamma_1 \le \left|\lambda_i(D^2(t) + D(t)A(t)J_f(X) - B(t)C(t)J_g(X) + C(t)J_g(X))\right| \le \gamma_2;$$

(iii)
$$-\gamma_3 \le \lambda_i(J_g(X)J_f(X)) \le -\gamma_4, -\Delta_3 \le \lambda_i(-B(t)D(t)) \le -\delta_3$$
where $J_f(X), J_g(X)$ *denote the Jacobian matrices* $\frac{\partial f_i}{\partial x_i}, \frac{\partial g_i}{\partial x_i}$ *of* $F(X)$ *and* $G(X)$, *respectively,*

(iv)
$$\lambda_i(A(t)) \le 1, \lambda_i(C(t)) \le 1, \lambda_i(B'(t)C(t)) \le 1, -\alpha_2 \le \lambda_i(D(t)) \le -\beta_2, \alpha_0 \le \lambda_i(D'(t)) \le \beta_0,$$
$$\alpha_1 \le \lambda_i(B'(t)) \le \beta_1, -\gamma_* - \gamma_4 - \delta_3 - \beta_2\beta_0 + \frac{\Delta_1\beta_0}{2} + \frac{\alpha_1\beta_2}{2} \le -\alpha,$$
$$\frac{\Delta_1\beta_0}{2} + \frac{\alpha_1\beta_2}{2} + \frac{1}{2}\beta_1 - \delta_1\beta_1 \le -\beta, |\lambda_i(D(t)D'(t))| \le \pi_1,$$
$$|\lambda_i(D(t)B'(t))| \le \pi_2, |\lambda_i(B(t)D'(t))| \le \pi_3,$$
$$|\lambda_i(B(t)B'(t))| \le \pi_4, \mu_1 \le \lambda_i(J_g(X)) \le \mu_2, i = 1, 2, ..., n.$$

(v) *The matrices* $B(t)$, $D(t)$ *and* $J_f(X)$ *are symmetric and negative definite while the matrices* $C(t)$, $D'(t)$ *and* $J_g(X)$ *are symmetric and positive definite;*

(vi) *The matrix* $B(t)$ *commutes with matrix* $D(t)$, *and the Jacobian matrices* J_g *and* J_f *also commute with each other;*

(vii) *The matrix* $\{D(.)J_f(X) - B(t)C(t)J_g(X)\}$ *is positive definite;*

(viii) *The matrix*
$$\{B(t)C(t)J_g(X_2) - D(.)J_f(X_2)\}\{D(t) + A(t)J_f(X_1)\}$$
is positive definite for arbitrary X_1, $X_2 \in R^n$.

Then the trivial solution of system (2) *is uniformly-asymptotically stable and the solution of system satisfy*
$$\|X(t)\| \to 0, \left\|\dot{X}(t)\right\| \to 0 \text{ as } t \to \infty.$$

Let us denote $V(t, X, Y)$ *by* $V(.)$, $\|X\| + \|Y\|$ *by* $\|Z\|$, $\|X\|^2 + \|Y\|^2$ *by* $\|T\|^2$, $D(t)A(t)$ *by* $D(.)$ *and* $\frac{dV(t,X,Y)}{dt}$ *by* $\dot{V}(.)$.

Theorem 2.2. *Let the all conditions of Theorem 2.1 hold. In addition, we assume the following condition hold:*

(v)
$$\|P_1(.)\| \le \delta_4 + \delta_5\|Z\|, \text{ and } \|P_2(.)\| \le \alpha_3 + \alpha_4\|Z\|,$$

where $\delta_4, \delta_5, \alpha_3$ and α_4 are some positive constants. Then, all solutions of system (2) are bounded, uniformly bounded and uniform-ultimately bounded.

Theorem 2.3. *Let the all conditions of Theorem 2.1 hold. In addition, we assume the following condition holds:*

(vi) $\|P_1(.)\| \leq \theta_1(t) + \theta_2(t)\|Z\|$ *and* $\|P_2(.)\| \leq \varphi_1(t) + \varphi_2(t)\|Z\|$ *for all* $t \geq 0$, $\max\theta_i(t) < \infty$, $\max\varphi_i(t) < \infty$ *and* $\theta_1(t),\ \theta_2(t),\ \varphi_1(t),\ \varphi_2(t) \in L^1(0,\infty)$, *where* $L^1(0,\infty)$ *is the space of integrable functions in sense of Lebesgue .* *Then, any solution* $(X(t), Y(t))$ *of system* (2) *with initial condition*

$$X(0) = X_0,\ Y(0) = Y_0$$

satisfies

$$\|X(t)\| \leq K, \|Y(t)\| \leq K$$

for all $t \geq 0$, *where* $K > 0$ *is a constant depending on* $B(t)$, $D(t)$, $\theta_1(t)$, $\phi_1(t)$, $\theta_2(t), \phi_2(t), t_0, X_0, Y_0$ *as well as on the functions* $P_1(.)$ *and* $P_2(.)$.

The main tool to be used in proving these theorems is the following Lyapunov function defined by

$$2V(.) =\|D(t)X - B(t)Y\|^2 + 2\int_0^1 \langle D(t)F(sX) - B(t)C(t)G(sX), X\rangle\, ds + 2\int_0^1 \langle C(t)G(sX), X\rangle\, ds - \langle B(t)Y, Y\rangle. \tag{3}$$

Before the proofs of Theorem 2.1, Theorem 2.2 and Theorem 2.3, we need the following lemmas, which will be used in the proof of these theorems.

Lemma 2.1. *Suppose that the conditions of Theorem 2.1 are satisfied. Then there exist positive constants, say* K_1, K_2 *and* K_3, *such that the Lyapunov function* V *of* (3) *satisfies*

$$K_1\|T\|^2 \leq 2V(.) \leq K_2\|T\|^2,$$
$$V(.) \to +\infty \text{ as } \|T\|^2 \to \infty$$

and

$$\frac{d}{dt}V(.) \leq -K_3\|T\|^2,$$

for all $X,\ Y \in R^n$.

Proof. It is obvious that $V(t, 0, 0) = 0$. Applying the conditions of Lemma 2.1, it follows that

$$\begin{aligned}2V(.) =&\|D(t)X - B(t)Y\|^2 + 2\int_0^1\int_0^1 \langle\{D(.)J_f(s_1s_2X) - B(t)C(t)J_g(s_1s_2X)\}X, X\rangle s_1ds_1ds_2\\ &+ 2\int_0^1\int_0^1 \langle C(t)J_g(s_1s_2X)X, X\rangle s_1ds_1ds_2 - \langle B(t)Y, Y\rangle\\ \geq& 2\int_0^1\int_0^1 \langle\{D(.)J_f(s_1s_2X) - B(t)C(t)J_g(s_1s_2X) + C(t)J_g(s_1s_2X)\}X, X\rangle s_1ds_1ds_2.\end{aligned}$$

By using the conditions of Theorem 2.1, we have

$$\langle \{D(.)J_f(s_1s_2X) - B(t)C(t)J_g(s_1s_2X) + C(t)J_g(s_1s_2X)X, X\}\rangle \geq \delta_0 \|X\|^2$$

and

$$-\langle B(t)Y, Y\rangle \geq \delta_1 \|Y\|^2.$$

Thereby, we see that

$$2\int_0^1\int_0^1 \langle \{D(.)J_f(s_1s_2X) - B(t)C(t)J_g(s_1s_2X) + C(t)J_g(s_1s_2X)\}X, X\rangle s_1 ds_1 ds_2 \geq \delta_0 \|X\|^2.$$

Hence, we get

$$2V(.) \geq \delta_0\|X\|^2 + \delta_1\|Y\|^2 \geq K_1\|Z\|^2 \quad \text{for all } X, Y \in R^n, \tag{4}$$

where $K_1 = \min\{\delta_0, \delta_1\}$.

Next, it follows from (4) that $V(.) = 0$ if and only if $\|Z\|^2 = 0$, and $V(.) > 0$ if $\|Z\|^2 \neq 0$, which implies that

$$V(.) \to \infty \text{ as } \|Z\|^2 \to \infty.$$

In a similar way, by Theorem 2.1 we have

$$\left\langle \left\{D(.)J_f(s_1s_2X) - B(t)C(t)J_g(s_1s_2X) + C(t)J_g(s_1s_2X)X, X\right\}\right\rangle \leq \Delta_0 \|X\|^2,$$

$$-\langle B(t)Y, Y\rangle \leq \Delta_1 \|Y\|^2$$

and

$$\|D(t)X - B(t)Y\|^2 \leq \Delta_*\|Z\|^2.$$

Thereby, we get

$$2\int_0^1\int_0^1 \langle \{D(.)J_f(s_1s_2X) - B(t)C(t)J_g(s_1s_2X) + C(t)J_g(s_1s_2X)\}X, X\rangle s_1 ds_1 ds_2 \leq \Delta_0 \|X\|^2.$$

Hence, it is clear that

$$2V(.) \leq \Delta_*\|Z\|^2 + \Delta_0\|X\|^2 + \Delta_1\|Y\|^2.$$

Let $K_2 = \max\{\Delta_* + \Delta_0, \Delta_* + \Delta_1\}$. Then,

$$2V(.) \leq K_2\|Z\|^2, \tag{5}$$

for all $X, Y \in R^n$. Consequently, from the inequalities (4) and (5), we have

$$K_1\|Z\|^2 \leq 2V(.) \leq K_2\|Z\|^2.$$

Next, we calculate the derivative of V with respect to t along the solutions of system (2). Then, the derivative of the Lyapunov function V along solutions of system (2) is obtained as follows:

$$\begin{aligned}\frac{d}{dt}V(.) = &\langle D(t)X + A(t)F(X), D(t)A(t)F(X) - B(t)C(t)G(X)\rangle \\ &+ \langle C(t)G(X), A(t)F(X)\rangle - \langle B(t)D(t)Y, Y\rangle + \langle D(t)X, D'(t)X\rangle \\ &- \langle B(t)Y, D'(t)X\rangle + \langle B(t)Y, B'(t)Y\rangle\end{aligned}$$

$$
\begin{aligned}
& - \langle B'(t)Y, D(t)X\rangle + \frac{1}{2}\langle B'(t)Y, Y\rangle \\
= & \int_0^1\int_0^1 \langle \{D(t) + A(t)J_f(s_1X)\} \\
& \times \{D(.)J_f(s_2X) - B(t)C(t)J_g(s_2X)\} X, X\rangle ds_1 ds_2 \\
& + \int_0^1\int_0^1 \langle C(t)J_g(s_1X)X, A(t)J_f(s_2X)X\rangle \, ds_1 ds_2 \\
& - \langle B(t)D(t)Y, Y\rangle + \langle D(t)X, D'(t)X\rangle - \langle B(t)Y, D'(t)X\rangle + \langle B(t)Y, B'(t)Y\rangle \\
& - \langle B'(t)Y, D(t)X\rangle + \frac{1}{2}\langle B'(t)Y, Y\rangle .
\end{aligned}
$$

By the conditions of Theorem 2.1, we obtain

$$
\begin{aligned}
\frac{d}{dt}V(.) \le & [-\gamma_* - \gamma_4 - \delta_3 - \beta_2\beta_0 + \frac{\Delta_1\beta_0}{2} + \frac{\alpha_1\beta_2}{2}]\|X\|^2 \\
& + [\frac{\Delta_1\beta_0}{2} + \frac{\alpha_1\beta_2}{2} + \frac{1}{2}\beta_1 - \delta_1\beta_1]\|Y\|^2 \\
= & -\alpha\|X\|^2 - \beta\|Y\|^2.
\end{aligned}
$$

Thus, there exists a constant $K_3 = \min\{\alpha, \beta\} > 0$ such that

$$\dot{V} \le -K_3\|Z\|^2,$$

for all $X, Y \in R^n$. This completes the proof of Lemma 2.1. □

Let $P_1(.) \neq 0$ and $P_2(.) \neq 0$.

Lemma 2.2. *We assume that the conditions Theorem 2.2 hold. Then there exist positive constants K_6 and K_7 such that*

$$\frac{d}{dt}V(.) \le -K_6\|T\|^2 + K_8\sqrt{\|T\|^2}\{\|P_1(.)\| + \|P_2(.)\|\}$$

for all $X, Y \in R^n$.

Proof. Following the calculations in the proof of Lemma 2.1, we obtain

$$
\begin{aligned}
\frac{d}{dt}V(.) = & \langle D(t)X, D'(t)X + D(.)F(X) + D(t)P_1(.), \\
& - B'(t)Y - B(t)C(t)G(X) - B(t)P_2(.)\rangle \\
& - \langle B(t)Y, D'(t)X + D(t)P_1(.) - B'(t)Y - B(t)P_2(.)\rangle \\
& + \langle D(.)F(X), A(t)F(X) + P_1(.))\rangle \\
& - \langle B(t)C(t)G(X), A(t)F(X) + P_1(.)\rangle + \langle C(t)G(X), A(t)F(X) + P_1(.)\rangle \\
& - \langle B(t)Y, D(t)Y + P_2(.)\rangle - \frac{1}{2}\langle B'(t)Y, C(t)G(X) + D(t)Y + P_2(.)\rangle \\
= & \langle D(t)X + A(t)F(X), D(.)F(X) - B(t)C(t)G(X)\rangle \\
& + \langle C(t)G(X), A(t)F(X)\rangle - \langle B(t)Y, D(t)Y\rangle \\
& + \langle D^2(t)X + D(.)F(X) - B(t)C(t)G(X) \\
& + C(t)G(X) - B(t)D(t)Y, P_1(.)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&-\left\langle B(t)D(t)X + B(t)Y - B^2(t)Y - \frac{1}{2}B'(t)Y, P_2(.)\right\rangle \\
&+\langle D(t)X, D'(t)X - B'(t)Y\rangle - \langle B(t)Y, D'(t)X - B'(t)Y\rangle \\
&-\frac{1}{2}\langle B'(t)Y, C(t)G(X) + D(t)Y\rangle \\
=&\int_0^1\int_0^1 \langle D(t)X + A(t)J_f(s_1X)X, D(.)J_f(s_2X)X - B(t)C(t)J_g(s_2X)X\rangle\, ds_1 ds_2 \\
&+\int_0^1\int_0^1 \langle C(t)J_g(s_1X)X, A(t)J_f(s_2X)X\rangle\, ds_1 ds_2 - \langle B(t)Y, D(t)Y\rangle \\
&+\int_0^1 \left\langle \left\{D^2(t) + D(.)J_f(s_1X) - B(t)C(t)J_g(s_1X) + C(t)J_g(s_1X)\right\} X, P_1(.)\right\rangle ds_1 \\
&-\int_0^1 \langle B(t)D(t)Y, P_1(.)\rangle\, ds_1 + \left\langle B^2(t)Y - B(t)D(t)X - B(t)Y + \frac{1}{2}B'(t)Y, P_2(.)\right\rangle \\
&+\langle D(t)X, D'(t)X - B'(t)Y\rangle - \langle B(t)Y, D'(t)X - B'(t)Y\rangle \\
&-\frac{1}{2}\langle B'(t)Y, C(t)G(X) + D(t)Y\rangle \\
=&\int_0^1\int_0^1 \langle \{D(t) + A(t)J_f(s_1X)\}\{D(.)J_f(s_2X) - B(t)C(t)J_g(s_2X)\} X, X\rangle ds_1 ds_2 \\
&+\int_0^1 \left\langle \left\{D^2(t) + D(.)J_f(s_1X) - B(t)C(t)J_g(s_1X) + C(t)J_g(s_1X)\right\} X, P_1(.)\right\rangle ds_1 \\
&-\int_0^1 \langle B(t)D(t)Y, P_1(.)\rangle\, ds_1 + \int_0^1\int_0^1 \langle C(t)J_g(s_1X)X, A(t)J_f(s_2X)X\rangle\, ds_1 ds_2 \\
&+\left\langle B^2(t)Y - B(t)D(t)X - B(t)Y + \frac{1}{2}B'(t)Y, P_2(.)\right\rangle \\
&+\langle D(t)X, D'(t)X - B'(t)Y\rangle - \langle B(t)Y, D'(t)X - B'(t)Y\rangle \\
&-\frac{1}{2}\langle B'(t)Y, C(t)G(X) + D(t)Y\rangle - \langle B(t)Y, D(t)Y\rangle \\
\leq& -\gamma_*\|X\|^2 - \delta_3\|Y\|^2 - \gamma_4\|X\|^2 \\
&+\int_0^1 \left\langle \left\{D^2(t) + D(.)J_f(s_1X) - B(t)C(t)J_g(s_1X) + C(t)J_g(s_1X)\right\} X, P_1(.)\right\rangle ds_1 \\
&-\int_0^1 \langle B(t)D(t)Y, P_1(.)\rangle\, ds_1 + \langle D(t)X, D'(t)X - B'(t)Y\rangle \\
&+\left\langle B^2(t)Y - B(t)D(t)X - B(t)Y + \frac{1}{2}B'(t)Y, P_2(.)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&- \langle B(t)Y, D'(t)X - B'(t)Y \rangle - \frac{1}{2} \langle B'(t)Y, C(t)G(X) + D(t)Y \rangle \\
&\leq \left[-K_6 + \pi_1 + 2^{-1}(\pi_2 + \pi_3) + 4^{-1}\mu_2\right] \|X\|^2 + \left[-K_6 + \pi_2 + 2^{-1}\pi_3 + \pi_4 + 4^{-1}\mu_2\right] \|Y\|^2 \\
&\quad + \{\Delta_3 \|X\| + \Delta_2 \|Y\|\} \|P_2(.)\| + \{\gamma_2 \|X\| + \Delta_3 \|Y\|\} \|P_1(.)\| \\
&\leq - K_* \|T\|^2 + K_7 \|Z\| \{\|P_1(.)\| + P_2(.)\},
\end{aligned}
$$

where

$$K_6 = \min\{\gamma_*, \delta_3, \gamma_4\}, K_7 = \max\{\gamma_2, \Delta_2, \Delta_3\},$$

$$K_* = \max\left\{\left(-K_6 + \pi_1 + 2^{-1}(\pi_2 + \pi_3) + 4^{-1}\mu_2\right), \left(-K_6 + \pi_2 + 2^{-1}\pi_3 + \pi_4 + 4^{-1}\mu_2\right)\right\}.$$

However, since

$$\|Z\| \leq \sqrt{2\|T\|^2},$$

we have

$$\frac{d}{dt}V(.) \leq -K_6\|T\|^2 + K_8\sqrt{\|T\|^2}\{\|P_1(.)\| + \|P_2(.)\|\},$$

where $K_8 = \sqrt{2}K_7$, for all $t \geq 0$. This completes the proof of Lemma 2.2. □

Let $P_1(.) \equiv 0$ and $P_2(.) \equiv 0$ in (1).

Proof of Theorem 2.1. The proof is similar to that of Adeyanju et al. [6, Theorem 3.1]. We leave out the proof.

Proof of Theorem 2.2. The proof is similar to that of Adeyanju et al. [6, Theorem 3.2]. We will ignore the proof.

Proof of Theorem 2.3. The proof is similar to that of Adeyanju et al. [6, Theorem 3.3]. We will not provide the proof.

We will now give two examples to show that the conditions of the given theorems can be hold in particular cases.

Example 1. *For the case $n = 2$, we consider the following system as a particular case of (2):*

$$A(t) = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix}, B(t) = \begin{pmatrix} -0.21 & 0 \\ 0 & -1.1 \end{pmatrix}, C(t) = \begin{pmatrix} 0.101 & 0 \\ 0 & 0.102 \end{pmatrix},$$

$$D(t) = \begin{pmatrix} -0.011 & 0 \\ 0 & -0.0011 \end{pmatrix}, F(X) = \begin{pmatrix} \arctan x_1 - 1.02x_1 \\ -0.101x_2 \end{pmatrix},$$

$$G(X) = \begin{pmatrix} -\cos x_1 + 1.2x_1 \\ -\cos x_2 + 1.2x_2 \end{pmatrix},$$

$$J_f(X) = \begin{pmatrix} \frac{1}{1+x_1^2} - 1.02 & 0 \\ 0 & -1.101 \end{pmatrix},$$

$$J_g(X) = \begin{pmatrix} \sin x_1 + 1,2 & 0 \\ 0 & \sin x_2 + 1.2 \end{pmatrix},$$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \dot{X} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}, Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \dot{Y} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix}.$$

Then, we have the following system of scalar differential equations:

$$\dot{x}_1 = 0.1 \arctan x_1 - 0.102 - 0.21y_1,$$

$$\dot{x}_2 = 0.0202x_2 - 1.1y_2,$$

$$\dot{y}_1 = -0.101 \cos x_1 + 0.1212 - 0.0011y_1,$$

$$\dot{y}_2 = -0.102 \cos x_2 + 0.1224 - 0.0011y_2.$$

Hence, we have

$$D(t)A(t)J_f(X) - B(t)C(t)J_g(X)$$
$$= \begin{pmatrix} \frac{-0.0011}{1+x_1^2} + 0.02121\sin x_1 + 0.026574 & 0 \\ 0 & 0.1122\sin x_2 + 0.13488222 \end{pmatrix}$$

and

$$\{B(t)C(t)J_g(X) - D(t)A(t)J_g(X)\}\{D(t) + A(t)J_f(X)\}$$
$$= \begin{pmatrix} [\frac{0.0011}{1+x_1^2} - 0.02121\sin x_1 - 0.026574][\frac{0.1}{1+x_1^2} - 0.113] & 0 \\ 0 & 0.02482986\sin x_2 + 0.0298494353 \end{pmatrix},$$

where

$$\delta_a = 0.1, \Delta_a = 0.2, \delta_b = -1.1, \Delta_b = -0,21, \delta_c = 0.101, \Delta_c = 0.102,$$
$$\delta_d = -0.011, \Delta_d = -0.0011, \delta_f = -1.02, \Delta_f = -0.02,$$
$$\delta_g = 1.2, \Delta_g = 2,2, \delta_{a^*} = 0.004264, \Delta_{a^*} = 0.24708222,$$
$$\delta_{b^*} = 0.0002959762, \Delta_{b^*} = 0.0546792953.$$

Hence, the conditions of Theorem 2.1 hold.

Let $P_1(.) \neq 0$ and $P_2(.) \neq 0$.

Example 2. *We consider the Example 1, which also includes the following terms additionally:*

$$P_1(.) = \frac{1}{\left[2t^2 + (x_1+y_1)^2 + (x_2+y_2)^2 + 2\right]^2} \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix},$$
$$P_2(.) = \frac{1}{[e^{2t}+2]^2} \begin{pmatrix} x_1 + y_1\cos x_1 \\ x_2 + y_2\cos x_2 \end{pmatrix}.$$

No need to reconsider the discussion in Example 1. In addition the outcomes of Example 1, we have the following relations:

$$\begin{aligned} \|P_1(.)\| \leq & \frac{1}{\left[2t^2 + (x_1+y_1)^2 + (x_2+y_2)^2 + 2\right]} \sqrt{(x_1+y_1)^2 + (x_2+y_2)^2} \\ \leq & \frac{\sqrt{2}}{(t^2+1)} \sqrt{x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2} \\ \leq & \frac{\sqrt{2}}{(t^2+1)} \sqrt{\|X\|^2 + \|Y\|^2 + 2} \\ \leq & \frac{\sqrt{2}}{(t^2+1)} \left\{\|X\| + \|Y\| + \sqrt{2}\right\} \\ \leq & \frac{2}{(t^2+1)} + \frac{\sqrt{2}}{(t^2+1)} \{\|X\| + \|Y\|\} \\ = & \theta_1(t) + \theta_2(t)\{\|X\| + \|Y\|\} \\ \leq & 2 + \sqrt{2}\{\|X\| + \|Y\|\}, \end{aligned}$$

where $\theta_1(t) = \frac{2}{t^2+1} \leq 2 \leq \delta_0$ *and* $\theta_2(t) = \frac{\sqrt{2}}{t^2+1} \leq \sqrt{2} \leq \delta_1$;

Similarly, we have

$$\|P_2(.)\| \leq \frac{1}{[e^{2t}+2]} \sqrt{(x_1 + y_1\cos x_1)^2 + (x_2 + y_2\cos x_2)^2}$$

$$
\begin{aligned}
&\leq \frac{1}{(e^{2t}+2)}\sqrt{2(x_1^2+y_1^2+x_2^2+y_2^2)+4} \\
&\leq \frac{\sqrt{2}}{(e^{2t}+2)}\sqrt{x_1^2+y_1^2+x_2^2+y_2^2+2} \\
&\leq \frac{\sqrt{2}}{(e^{2t}+2)}\sqrt{\|X\|^2+\|Y\|^2+2} \\
&\leq \frac{\sqrt{2}}{(e^{2t}+2)}\left\{\|X\|+\|Y\|+\sqrt{2}\right\} \\
&\leq \frac{2}{(e^{2t}+2)}+\frac{\sqrt{2}}{(e^{2t}+2)}\left\{\|X\|+\|Y\|\right\} \\
&= \varphi_1(t)+\varphi_2(t)\left\{\|X\|+\|Y\|\right\} \\
&\leq 2+\sqrt{2}\left\{\|X\|+\|Y\|\right\},
\end{aligned}
$$

where $\theta_1(t)=\frac{2}{e^{2t}+2}\leq 2\leq\alpha_0$ *and* $\theta_2(t)=\frac{\sqrt{2}}{e^{2t}+2}\leq\sqrt{2}\leq\alpha_1$.

Thus, all the conditions of Theorems 2.1 and Theorem 2.2 are satisfied by this example.

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References

[1] D. O. Adams, A. A. Adeyanju , M. O. Omeike, *Asymptotic stability and boundedness criteria for a certain second order nonlinear differential equations*, J. Nigerian Math. Soc., **43**(1) (2024), 81-88.

[2] A. A. Adeyanju, *Stability and boundedness criteria of solutions of a certain system of second order differential equations*, Ann. Univ. Ferrara Sez. VII Sci. Mat., **69**(1) (2023), 81-93.

[3] A.A. Adeyanju, D.O. Adams, *Some new results on the stability and boundedness of solutions of certain class of second order vector differential equations*, International Journal of Mathematical Analysis and Optimization: Theory and Applications., **7**(1) (2021), 108-115.

[4] A. A. Adeyanju, M. O. Omeike, O. J. Adeniran, B. S. Badmus, *Stability and boundedness of solutions of certain Aizermann differential equations*, Acta Univ. Apulensis Math., Inform., (72) (2022), 131-153.

[5] A. A. Adeyanju, C.Tunç, *Uniform-ultimate boundedness of solutions to vector Lienard equation with delay*, Ann. Univ. Ferrara Sez. VII Sci. Mat., **69** (2) (2023), 605–614.

[6] A. A. Adeyanju, M. O. Omeike, J. O. Adeniran, B. S. Badmus, *On the stability and boundedness of solutions of Aizermann vector differential equations*, J. Nigerian Math. Soc., **42**(3) (2023), 169-179.

[7] A.A. Adeyanju, A. Adetunji, D.O. Adams, *On the stability and boundedness of solutions to certain second order differential equation*, Differ. Uravn. Protsessy Upr., (3) (2023), 59–70.

[8] S. Ahmad, M. Rama Mohana Rao, *Theory of ordinary differential equations. With applications in biology and engineering*, Affiliated East-West Press Pvt. Ltd., New Delhi, (1999).

[9] L. Berezansky, E. Braverman, *Exponential stability for a system of second and first order delay differential equations*, Appl. Math. Lett., **132** (2022), Paper No. 108127, 7 pp.

[10] T. A. Burton, *Stability and periodic solutions of ordinary and functional differential equations*, Corrected version of the 1985 original. Dover Publications, Inc., Mineola, NY, (2005).

[11] K. T. Chau, *Applications of differential equations in engineering and mechanics*, CRC Press, Boca Raton, FL, (2019).

[12] J. O. C. Ezeilo, *On the convergence of solutions of certain systems of second order differential equations*, Ann. Mat. Pura Appl., **72** (4) (1966), 239–252.

[13] J. O. C. Ezeilo, *On the stability of solutions of certain systems of ordinary differential equations*, Ann. Mat. Pura Appl., **73** (4) (1966), 17–26.

[14] M. Gözen, *New Qualitative Outcomes for Ordinary Differential Systems of Second Order*, Contemp. Math., (4)(2023), 1210–1221.

[15] M. Gözen, *Boundedness of Vector Lineard Equation with Multiple Variable Delay*, Mathematics, **12**(769)(2024).

[16] M. Gözen, C. Tunç, *Stability in functional integro-differential equations of second order with variable delay*, J. Math. Fundam. Sci., **49** (1) (2017), 66–89.

[17] J. R. Graef, *On the generalized Liénard equation with negative damping*, J. Differential Equations, **12**(1972), 34–62.

[18] J. R. Graef, P. W. Spikes, *Boundedness and convergence to zero of solutions of a forced second-order nonlinear differential equation*, J. Math. Anal. Appl., **62** (2) (1978), 295–309.

[19] S. B. Hsu, *Ordinary differential equations with applications*, Series on Applied Mathematics, 16. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, (2006).

[20] J. Jiang, *The global stability of a class of second order differential equations*, Nonlinear Anal, **28**(5) (1997), 855–870.

[21] Z. Jin, *Boundedness and convergence of solutions of a second-order nonlinear differential system*, J. Math. Anal. Appl., **256** (2)(2001), 360–374.

[22] S. Kulcsar, *Boundedness, convergence and global stability of solutions of a nonlinear differential equation of the second order*, Publ. Math. Debrecen, **37**(3-4) (1990), 193–201.

[23] J. P. LaSalle, S. Lefschetz, *Stability by Liapunov's direct method, with applications*, Mathematics in Science and Engineering, Vol. 4. Academic Press, New York-London, (1961).

[24] C. Tunç, O. Tunç, Y. Wang, J. C. Yao, *Qualitative analyses of differential systems with time-varying delays via Lyapunov–Krasovskii* approach. Mathematics, (2021) **9**(11), 1196.

[25] N. Nakanishi, K. Seto, *Differential equations and their applications—analysis from a physicist's viewpoint*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, (2022).

[26] A. L. Olutimo, *On the convergence behaviour of solutions of certain system of second order nonlinear delay differential equations*, J. Nigerian Math. Soc., **42**(2) (2023).

[27] M. O. Omeike, *Further stability criteria for certain second-order delay differential equations with mixed coefficients*, J. Nigerian Math. Soc.,**42**(1) (2023), 36–48.

[28] M. O. Omeike, O. O. Oyetunde, A. L. Olutimo, *Boundedness of solutions of certain system of second-order ordinary differential equations*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math., **53**(1) (2014),107–115.

[29] C. X. Qian, *On global asymptotic stability of second order nonlinear differential systems*, Nonlinear Anal, **22**(7) (1994), 823–833.

[30] J. Sugie, *On the boundedness of solutions of the generalized Lienard equation without the signum condition*, Nonlinear Anal.,**11**(12)(1987),1391–1397.

[31] H. O. Tejumola, *Boundedness criteria for solutions of some second-order differential equations*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat., **50**(8) (1971), 432–437.

[32] H. O. Tejumola, *On a Lienard type matrix differential equation.* Atti Accad, Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat., **60**(8)(2) (1976), 100–107.

[33] C. Tunç, *Some new stability and boundedness results on the solutions of the nonlinear vector differential equations of second order.* Iran. J. Sci. Technol. Trans. A Sci., **30**(2) (2006), 213–221.

[34] O. Tunç, *On the behaviors of solutions of systems of non-linear differential equations with multiple constant delays*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM 115 (2021), (4), Paper No. 164, 22 pp.

[35] C. Tunç, M. Altun, *On the integrability of solutions of non-autonomous differential equations of second order with multiple variable deviating arguments*, J. Comput. Anal. Appl., **14**(5) (2012), 899–908.

[36] C. Tunc, Y. Dinç, *Qualitative properties of certain non-linear differential systems of second order.*, Journal of Taibah University for Science, **11**(2) (2017),359–366.

[37] C. Tunç, O. Tunç, *Qualitative analysis for a variable delay system of differential equations of second order.*, Journal of Taibah University for Science., **13**(1) (2019), 468–477.

[38] O. Tunç, C. Tunç, On the asymptotic stability of solutions of stochastic differential delay equations of second order., Journal of Taibah University for Science., **13**(1) (2019), 875–882.

[39] T. Yoshizawa, *Stability theory by Liapunov's second method.*,Publications of the Mathematical Society of Japan, 9. Mathematical Society of Japan, Tokyo, (1966).

[40] T. Yoshizawa, *Stability theory and the existence of periodic solutions and almost periodic solutions.*, Applied Mathematical Sciences, Vol. 14. Springer-Verlag, New York-Heidelberg, (1975).

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