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DIFFERENT FORMS OF CONVEXITY IN METRIC LINEAR SPACES

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Abstract. Various forms of convexity in normed linear spaces that fall between strict and uniform convexity, such as the $2R$ property, local uniform convexity, compact local uniform convexity, and mid-point local uniform convexity, have sparked considerable interest over the years and have been thoroughly investigated in the literature. The notion of strict convexity was extended to metric linear spaces by Albinus in 1968, and that of uniform convexity by Ahuja et al. in 1977. In this paper, we extend the other mentioned forms of convexity to metric linear spaces. We explore the metric linear spaces that have these properties and establish inter-relationships between these spaces. We also give a characterization of strictly convex metric linear spaces.

1. Introduction

The study of special normed linear spaces, in which closed unit balls are round in the sense that unit spheres include no nontrivial line segments, was initiated independently by Clarkson [5] and Krein (see [2]). They used the term 'strictly convex' for the normed linear spaces with this special property , and later, Day [6] referred to these normed linear spaces as 'rotund'. Clarkson was particularly interested in the uniform version of this property and initiated the study of uniformly convex normed linear spaces. Since then, the concept of rotundity has sparked considerable interest in various branches of mathematics. In particular, the convexity properties have been of great interest to those working in the geometry of Banach spaces, fixed point theory, approximation theory etc.

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Aside from strict and uniform convexity in normed linear spaces, numerous other forms of convexity that fall between these two have piqued the interest of researchers over the years. The $2R$ property, local uniform convexity, compact local uniform convexity, and mid-point local uniform convexity are just a few that have gained popularity among numerous researchers over the years (see [4, 6, 7, 8, 10, 11, 12, 13, 19, 20, 21, 22, 23, 24]).

The notions of strict convexity and uniform convexity were extended to spaces more general than normed linear spaces, viz. metric linear spaces, by Albinus [3] and Ahuja et al.[1], respectively, and subsequently, the study of these convexities in metric linear spaces was taken up by Sastry et al. [16, 17, 18], Vasile'v [23] and others. We recall here that a *metric linear space* is defined as a linear space X with a metric d in which addition and scalar multiplication are both continuous and the metric d is translation invariant. It is easy to see that if $(X, ||.||)$ is a normed linear space, it is also a metric linear space, with the metric function d defined on X as $d(x, y) = ||x - y||$. On the other hand, there are plenty of examples of metric linear spaces that are not normed linear spaces (see[14],[15]).

It is worth mentioning that in some of the scenarios, it becomes important, though challenging, to consider the spaces that are not normable. For example, some of the problems in approximation theory are better suited to non-normable spaces because, many times, a metric is a natural measure of error while a norm is not suitable. Moreover, analysts have always been naturally interested in exploring the properties of normed linear spaces that remain intact in more general settings. With these two objectives in mind, we extend to metric linear spaces the various notions of convexity that fall between strict and uniform convexity, such as the $2R$ property, local uniform convexity, compact local uniform convexity, and mid-point local uniform convexity. We investigate the metric linear spaces equipped with these types of convexity and determine their interrelationships.

In Section 2, we explore strict convexity, ball convexity, and mid-point convexity in metric linear spaces and give a characterization of strictly convex metric linear spaces amongst the metric linear spaces that have ball convexity. We also prove that in a metric linear space, a closed mid-point convex set is convex. In Section 3, we give a sequential criterion for uniform convexity in metric linear spaces. In Section 4, we extend and explore the above mentioned properties in metric linear spaces.

In this papaer, we shall only be dealing with real metric linear spaces. Throughout, A^c will stand for the set theoretic complement of a set A, the symbol $\mathbb{N}$ will stand for the set of natural numbers, $\mathbb{R}$ for the set of real numbers and $\mathbb{R}^+$ for the set of all non-negative real numbers. $B[0, r]$ will denote the closed ball of radius r around 0 and $B(0, r)$ will denote the open ball around 0 of radius r. The symbols S_X and B_X will denote the unit sphere and the closed unit ball in X, respectively . For any $x, y \in X$, we will denote the closed line segment joining x and y by $[x, y]$, and the open line segment joining x and y by (x, y).

2. Strict convexity, ball convexity and mid-point convexity

In this section, we discuss strict convexity, ball convexity and mid-point convexity in metric linear spaces. We recall that a normed linear space X is said to be *strictly convex* if $||\frac{x+y}{2}|| < 1$, whenever x and y are different points of the unit sphere S_X. This notion was extended to metric linear spaces as follows.

Definition 2.1. *A metric linear space (X, d) is said to be* strictly convex *if for any $r > 0$ and $x, y \in X$, $d(x, 0) \leq r$, $d(y, 0) \leq r$ imply $d(\frac{x+y}{2}, 0) < r$ unless $x = y$.*

In other words, a metric linear space (X, d) is strictly convex, if whenever x and y are two distinct points in any closed ball $B[0, r]$, then mid-point joining x and y is strictly inside the ball. The above formulation of strict convexity is due to Ahuja et al. [1]. Vasilev [23] named such spaces as *strongly convex* and used them to study approximative properties of sets in metric linear spaces.

According to the following finding by Sastry and Naidu [16], whenever x and y are two different points in the closed ball $B[0, r]$, then not only the mid-point but the entire open line segment joining x and y lies precisely inside the ball $B[0, r]$.

Theorem 2.2. *Let (X, d) be a metric linear space. Then the following are equivalent:*

(i) $r > 0, d(x, 0) = r = d(y, 0)$ and $x \neq y$ imply $B(0, r) \cap (x, y) \neq \emptyset$.
(ii) X is strictly convex.
(iii) $r > 0, x \neq y, x, y \in B[0, r]$ imply $(x, y) \subset B(0, r)$.

From Theorem 2.2, we obtain the following:

Corollary 2.3. *Let (X, d) be a metric linear space. Then the following are equivalent:*

(i) X is strictly convex.
(ii) For any x, y $(x \neq y) \in X$ with $d(x, 0) = r = d(y, 0)$, imply $d\left(\frac{x+y}{2}, 0\right) < r$, where $r > 0$ is any real number

Proof. (i) implies (ii) is clear from the definition of strict convexity. Now assume that (ii) holds. Then for any two distinct points x and y of X with $d(x, 0) = r = d(y, 0)$, we have $\frac{x+y}{2} \in B(0, r) \cap (x, y)$, implying that the space (X, d) satisfies condition (i) of Theorem 2.2 and hence X is strictly convex. □

It is easy to see that any closed or open ball B in a normed linear space X is a convex subset of the space in the sense that whenever $x, y \in B$, the line segment connecting x and y also lies in B, that is, $[x, y] \subseteq B$ for all $x, y \in B$. But this may not be so in a metric linear space, as can be seen from the following example by Sastry and Naidu [16].

Example 2.4. *Define $f : \mathbb{R}^+ \to \mathbb{R}^+$ by*

$$f(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1; \\ \frac{1}{2}(1 + \frac{1}{t}), & \text{if } t > 1. \end{cases}$$

and $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ by $d(x, y) = f(|x - y|)$. Then $(\mathbb{R}, d)$ is a metric linear space and for $\frac{1}{2} < r < 1$, $B(0, r)$ is not convex.

Perhaps this observation led Sastry and Naidu [17] to explore the metric linear spaces in which all balls (open or closed) are convex subsets of the space. They referred to such metric linear spaces as the spaces having *ball convexity.* According to Theorem 2.2, a strictly convex metric linear space has ball convexity, while a metric linear space with ball convexity may not be strictly convex, as demonstrated by the following example by Sastry and Naidu [16].

Example 2.5. *Define* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *by*

$$f(t) = \begin{cases} t, & \text{if } 0 \le t \le 1; \\ 1, & \text{if } 1 < t < 2; \\ \frac{t}{2}, & \text{if } t \ge 2. \end{cases}$$

and $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ *by* $d(x, y) = f(|x - y|)$. *Then* $(\mathbb{R}, d)$ *is a metric linear space and balls are convex in* $(\mathbb{R}, d)$ *but this is not a strictly convex metric linear space, as* $d(1, 0) = f(1) = 1;\ d(2, 0) = f(2) = 1$*; and* $d(\frac{1+2}{2}, 0) = d(\frac{3}{2}, 0) = f(\frac{3}{2}) = 1 \nless 1$.

We recall that a subset K of a metric linear space is said to be *mid-point convex* if $\frac{x+y}{2} \in K$, whenever $x, y \in K$. Clearly, every convex set is mid-point convex but converse is not true, the set $\mathbb{Q}$ of rational numbers being an easy example of a mid-point convex subset that is not a convex subset of $\mathbb{R}$. In the following result, we prove that for closed subsets of a metric linear space, mid-point convexity implies convexity. For normed linear spaces, the result is well known (see [9])

Proposition 2.6. *A closed mid-point convex subset of a metric linear space is convex.*

Proof. Suppose A is a closed mid-point convex subset of a metric linear space (X, d). We show that A is convex. Let $x, y \in A$. Consider the mapping $T : [0, 1] \to X$ defined by $T(\lambda) = \lambda x + (1 - \lambda)y$. By the continuity of addition and scalar multiplication in X, T is continuous. Let

$$U = \{\lambda \in [0, 1] : T(\lambda) \notin A\} = \{\lambda \in [0, 1] : T(\lambda) \in A^c\} = T^{-1}(A^c).$$

By the continuity of T, U is open in $[0, 1]$. If $U = \emptyset$, then $\lambda x + (1 - \lambda)y \in A$ for all $\lambda \in [0, 1]$, as desired. So, assume that $U \ne \emptyset$. As $x, y \in A$ and A is mid-point convex, $\frac{x+y}{2} = T(\frac{1}{2}) \in A$. Thus $\frac{1}{2} \notin U$, implying that U is a non-empty proper open subset of $[0, 1]$. Therefore, we can find distinct $\lambda_1, \lambda_2 \in [0, 1] \backslash U$ such that $(\lambda_1, \lambda_2) \subset U$. Then $\frac{\lambda_1+\lambda_2}{2} \in (\lambda_1, \lambda_2) \subset U$ implies that $T\left(\frac{\lambda_1+\lambda_2}{2}\right) \notin A$. But $T\left(\frac{\lambda_1+\lambda_2}{2}\right) = \left(\frac{\lambda_1+\lambda_2}{2}\right) x + \left[1 - \left(\frac{\lambda_1+\lambda_2}{2}\right)\right] y = \frac{[\lambda_1 x+(1-\lambda_1)y]+[\lambda_2 x+(1-\lambda_2)y]}{2} = \frac{T(\lambda_1)+T(\lambda_2)}{2}$, which is not possible as $T(\lambda_1), T(\lambda_2) \in A$ and therefore, by mid-point convexity of A, $\frac{T(\lambda_1)+T(\lambda_2)}{2} \in A$. Thus $U = \emptyset$ and hence A is convex. □

It is well known (see [11, p. 440]) that a normed linear space is strictly convex if and only if any two closed balls in X having disjoint interior, do not intersect at more than one point. We extend this result to metric linear spaces.

Theorem 2.7. *(i) In a strictly convex metric linear space, no two closed balls having disjoint interiors intersect at more than one point.*
(ii) A metric linear space (X, d) *with ball convexity is strictly convex if no two closed balls in* X *having disjoint interiors intersect at more than one point.*

Proof. (i) Assume that $B[x, r]$ and $B[y, s]$ are two closed balls in a strictly convex metric linear space (X, d) such that $B(x, r) \cap B(y, s) = \emptyset$. If possible, assume that a, b $(a \ne b) \in B[x, r] \cap B[y, s]$. Then $d(x, a) \le r,\ d(x, b) \le r$ and $d(y, a) \le s, d(y, b) \le s$. As d is translation invariant, we have $d(x - a, 0) \le r,\ d(x - b, 0) \le r$ and $d(y - b, 0) \le s,\ d(y - b, 0) \le s$. Now as $a \ne b$, $x - a \ne x - b$ and $y - a \ne y - b$, therefore by strict convexity of X, we have

$$d\left(\frac{x - a + x - b}{2}, 0\right) < r, \text{ that is, } d\left(x - \frac{a + b}{2}, 0\right) < r$$

$$\text{and } d\left(\frac{y-a+y-b}{2},0\right) < s, \text{ that is, } d\left(y-\frac{a+b}{2},0\right) < s.$$

Again by the translation invariance of d, we have $d\left(x, \frac{a+b}{2}\right) < r$ and $d\left(y, \frac{a+b}{2}\right) < s$, implying that $\frac{a+b}{2} \in B(x,r) \cap B(y,s) = \emptyset$, which is not possible. Thus $B[x,r]$ and $B[y,s]$ cannot intersect at more than one point.

(ii) Suppose to the contrary that X is not strictly convex. Then there exist $a, b \in X, (a \neq b)$ and $r > 0$ such that $d(a,0) = r = d(b,0)$ but $d(\frac{a+b}{2}, 0) \geq r$. Now as $a, b \in B[0,r]$ and $B[0,r]$ is convex, we must have $\frac{a+b}{2} \in B[0,r]$, that is, $d\left(\frac{a+b}{2}, 0\right) \leq r$. Thus $d\left(\frac{a+b}{2}, 0\right) = r$. We claim that $B(0,r) \cap B(a+b,r) = \emptyset$. If $z \in B(0,r) \cap B(a+b,r)$, then $d(z,0) < r$ and $d(a+b,z) < r$, that is, $d(z,0) < r$ and $d(a+b-z,0) < r$, implying that $z, a+b-z \in B(0,r)$. Now, as $B(0,r)$ is convex, we have $\frac{1}{2}z + \frac{1}{2}(a+b-z) \in B(0,r)$, that is, $\frac{a+b}{2} \in B(0,r)$, which is not true as $d\left(\frac{a+b}{2}, 0\right) = r$. Thus, $B(0,r) \cap B(a+b,r) = \emptyset$. But $d(a,0) = r = d(b,0)$, implying that $a, b \in B[0,r]$, and $d(a, a+b) = d(0,b) = r = d(0,a) = d(b, a+b)$, implying that $a, b \in B[a+b, r]$. Thus $a, b (a \neq b) \in B[0,r] \cap B[a+b,r]$ but $B(0,r) \cap B(a+b,r) = \emptyset$, which contradicts the hypothesis and hence the result follows. □

From Theorem 2.7, we obtain the following characterization of strictly convex metric linear spaces:

Theorem 2.8. *A metric linear space (X,d) with ball convexity is strictly convex if and only if no two closed balls in X having disjoint interiors intersect at more than one point.*

3. Uniform convexity

In this section, we discuss uniform convexity in metric linear spaces. We recall that a normed linear space X is called *uniformly convex* if for each $\epsilon > 0$, there exists some $\delta > 0$ such that $||\frac{1}{2}(x+y)|| \leq 1 - \delta$, for all $x, y \in S_X$ with $||x-y|| \geq \epsilon$. While strict convexity ensures that every non-trivial line segment with endpoints on the space's unit sphere has its mid-point only in the interior of the closed unit ball, uniform convexity measures how deep in the interior of the closed unit ball the mid-point of such a segment will lie if the segment has some minimum positive length. The notion of uniform convexity was extended to metric linear spaces by Ahuja et al.[1] and later, Sastry and Naidu [16, 17] discussed different forms of uniform convexity in metric linear spaces. Throughout, we shall consider the following definition of uniform convexity, which was referred to as U.C II by Sastry and Naidu [16].

Definition 3.1. *A metric linear space (X,d) is called* uniformly convex *if given $r > 0$, $\epsilon > 0$, there exists some $\delta > 0$ such that $d(x,0) \leq r, d(y,0) \leq r$ and $d(x,y) \geq \epsilon$ imply $d(\frac{x+y}{2}, 0) \leq r - \delta$.*

The following examples of uniformly convex metric linear spaces are due to Sastry et al.[18].

Example 3.2. *Define $f : \mathbb{R}^+ \to \mathbb{R}^+$ by $f(x) = \log(1+x)$ and $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ by $d(x,y) = f(|x-y|)$. Then $(\mathbb{R}, d)$ is a metric linear space, which is uniformly convex.*

Example 3.3. *Define* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *by*

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ \frac{1}{n+1} + \frac{t}{n(n+1)}, & \text{if } x = \frac{1+t}{2^{n+1}}, 0 \le t \le 1, n \in \mathbb{N}; \\ x + \frac{1}{2}, & \text{if } x \ge \frac{1}{2}. \end{cases}$$

and $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ *by* $d(x, y) = f(|x - y|)$*. Then* $(\mathbb{R}, d)$ *is a uniformly convex metric linear space.*

It is well known that a uniformly convex metric linear space is strictly convex [16]. It was established by Sastry and Naidu [16] that a metric linear space (X, d) for which d is a bounded metric on X cannot be uniformly convex. On the other hand, the following example by them is of bounded strictly convex metric linear space and therefore, it is an example of a strictly convex metric linear space that is not uniformly convex.

Example 3.4. *Define* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *by* $f(t) = \frac{t}{1+t}$ *and* $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ *by* $d(x, y) = f(|x - y|)$*. Then* $(\mathbb{R}, d)$ *is a strictly convex metric linear space and clearly* $d(x, y) \le 1$ *for all* $x, y \in \mathbb{R}$*, thus* $(\mathbb{R}, d)$ *cannot be uniformly convex.*

It is known (see [11, Proposition 5.2.8]) that a normed linear space $(X, ||.||)$ is uniformly convex if and only if whenever $\{x_n\}$ and $\{y_n\}$ are sequences in B_X such that $||\frac{1}{2}(x_n + y_n)|| \to 1$, then $||x_n - y_n|| \to 0$. We generalize this result to metric linear spaces.

Proposition 3.5. *Let* (X, d) *be a metric linear space. If* (X, d) *is uniformly convex, then for any two sequences* $\{x_n\}, \{y_n\}$ *in* X *satisfying* $d(x_n, 0) \le r, d(y_n, 0) \le r$*, and* $d\left(\frac{x_n+y_n}{2}, 0\right) \to r$*, where* $r > 0$ *is any real number, we have* $d(x_n, y_n) \to 0$*. The converse holds if* (X, d) *has ball convexity.*

Proof. Let (X, d) be a uniformly convex metric linear space. Assume to the contrary that for some $r > 0$, there are sequences $\{x_n\}, \{y_n\}$ satisfying $d(x_n, 0) \le r, d(y_n, 0) \le r$ for all n and $d\left(\frac{x_n+y_n}{2}, 0\right) \to r$ but $d(x_n, y_n)$ does not converge to 0. Then for some $\epsilon > 0, m \in \mathbb{N}$ and a subsequence $\{(x_{n_k}, y_{n_k})\}$ of the sequence $\{(x_n, y_n)\}$, we have $d(x_{n_k}, y_{n_k}) \ge \epsilon$ for all $k \ge m$. As (X, d) is uniformly convex, for the pair (ϵ, r), there is a $\delta > 0$ for which $d\left(\frac{x_{n_k}+y_{n_k}}{2}, 0\right) \le r - \delta$, but this contradicts the fact that $d\left(\frac{x_{n_k}+y_{n_k}}{2}, 0\right) \to r$.

Conversely, assume that (X, d) is a metric linear space with ball convexity for which the sequences mentioned in the statement have the required behaviour. If (X, d) were not uniformly convex, then there would exist a pair (ϵ, r) of positive real numbers and a pair of points, $x_n, y_n \in X$ for each n, satisfying $d(x_n, 0) \le r, d(y_n, 0) \le r, d(x_n, y_n) \ge \epsilon$ but $d\left(\frac{x_n+y_n}{2}, 0\right) > r - \frac{1}{n}$. By the convexity of the ball $B[0, r]$, we have $d\left(\frac{x_n+y_n}{2}, 0\right) \le r$, implying that $d\left(\frac{x_n+y_n}{2}, 0\right) \to r$. But, as $d(x_n, y_n) \ge \epsilon$ for all n, $d(x_n, y_n)$ does not converge to zero, which contradicts the hypothesis. Hence (X, d) must be uniformly convex. □

4. Some convexity conditions in-between strict and uniform convexity

We start off this section by discussing the $2R$ property, a weaker variant of uniform convexity in normed linear spaces. We recall that a normed linear space $(X, ||.||)$ is said to have the *2R property* [22] if any sequence $\{x_n\}$ in S_X satisfying

$\lim_{n,m\to\infty} ||\frac{x_n+x_m}{2}|| = 1$ is a Cauchy sequence. It is known that a uniformly convex normed linear space has the 2R property (see [19, p. 77]) and a normed linear space having the 2R property is strictly convex. On the other hand, Fan and Glicksberg [7] gave an example of a normed linear space with the $2R$ property that is not uniformly convex. Smith [20] gave an example of a strictly convex normed linear space which does not have the $2R$ property. Thus, the $2R$ property is a property strictly in-between the strict convexity and the uniform convexity. We extend the notion of the $2R$ property to metric linear spaces as follows.

Definition 4.1. *A metric linear space (X, d) is said to have the* 2R property *if for any positive real number r, any sequence $\{x_n\}$ in X satisfying $d(x_n, 0) = r$ and $\lim_{n,m\mapsto\infty} d\left(\frac{x_n+x_m}{2}, 0\right) = r$, is a Cauchy sequence.*

We show that, as in normed linear spaces, uniform convexity implies the $2R$ property in metric linear spaces also.

Proposition 4.2. *A uniformly convex metric linear space has 2R property*

Proof. Let (X, d) be a uniformly convex metric linear space. For a positive real number r, let $\{x_n\}$ be a sequence in X such that $d(x_n, 0) = r$ for all n and $\lim_{n,m\to\infty} d\left(\frac{x_n+x_m}{2}, 0\right) = r$. Let $\epsilon > 0$ be given. Since (X, d) is uniformly convex, for the pair (ϵ, r), there exists $\delta > 0$ such that $d(x, 0) \leq r, d(y, 0) \leq r, d(x, y) \geq \epsilon$ imply $d\left(\frac{x+y}{2}, 0\right) \leq r - \delta$. As $\lim_{n,m\to\infty} d\left(\frac{x_n+x_m}{2}, 0\right) = r$, there exists $k \in \mathbb{N}$ such that $r - \delta < d\left(\frac{x_n+x_m}{2}, 0\right) < r + \delta$ for all $n, m \geq k$. By the definition of δ, we have $d(x_n, x_m) < \epsilon$ for all $n, m \geq k$, so that $\{x_n\}$ is a Cauchy sequence. □

It was already noted that the ball convexity is always present in a normed linear space but not necessarily in a metric linear space, and that ball convexity follows from strict convexity in a metric linear space but not the other way around. We next show that in a metric linear space, ball convexity in the presence of the $2R$ property implies the strict convexity.

Proposition 4.3. *A metric linear space having ball convexity and $2R$ property is strictly convex.*

Proof. Let (X, d) be a metric linear space having both ball convexity as well as the $2R$ property. Suppose to the contrary that (X, d) is not strictly convex. Then by Corollary 2.3, there exists $r > 0$ and $x, y \in X$ with $x \neq y$ such that $d(x, 0) = r = d(y, 0)$ but $d\left(\frac{x+y}{2}, 0\right) \geq r$. Since $x, y \in B[0, r]$ and $B[0, r]$ is convex, we have $d\left(\frac{x+y}{2}, 0\right) \leq r$; therefore, $d\left(\frac{x+y}{2}, 0\right) = r$. Let $\{x_n\}$ be the sequence in X defined by

$$x_n = \begin{cases} x, & \text{if } n \text{ is even;} \\ y, & \text{if } n \text{ is odd.} \end{cases}$$

Then $d(x_n, 0) = r$ and $d\left(\frac{x_n+x_m}{2}, 0\right) = r$ for all $n, m \in \mathbb{N}$. But $d(x_{2n},\ x_{2n-1}) = d(x, y) > \epsilon = \frac{d(x,y)}{2}$ for all n. Thus, $\{x_n\}$ is not a Cauchy sequence, which contradicts that X has the $2R$ property. □

We are unsure if a metric linear space with the $2R$ property has ball convexity, and we leave it as an open question.

Question 4.4. *Is ball convexity present in a metric linear space with the $2R$ property?*

A few other forms of convexity that have been studied in case of normed linear spaces are ‘local uniform convexity’, ‘compact local uniform convexity’, ‘mid-point local uniform convexity’([11, Chapter 5] and [24]). For brevity, these properties are referred to as LUR, CLUR, and MLUR respectively. Lovaglia [10] called a normed linear space LUR, if for each $\epsilon > 0$ and $x \in X$ with $||x|| = 1$, there exists $\delta(\epsilon, x) > 0$ such that $||\frac{1}{2}(x+y)|| \leq 1 - \delta$ for all $y \in S_X$ with $||x - y|| \geq \epsilon$. It is known that every uniformly convex normed linear space is LUR and every LUR normed linear space is strictly convex (see [11, page 460]). Smith[21] proved that neither of the two reverse implications is true, thereby implying that LUR property is strictly between strict convexity and uniform convexity. We now extend the notion of LUR to metric linear spaces as follows.

Definition 4.5. *A metric linear space (X, d) is called* LUR space *if given $\epsilon > 0$ and $x \in X$, there exists $\delta(\epsilon, x)$ such that $d\left(\frac{x+y}{2}, 0\right) \leq r - \delta(\epsilon, x)$, whenever $d(x, y) \geq \epsilon$ and $d(y, 0) \leq r$, where $r = d(x, 0)$.*

It is clear from the definition that every uniformly convex metric linear space is an LUR space. We now show that any LUR metric linear space is strictly convex, just like in the case of normed linear spaces.

Proposition 4.6. *Every LUR metric linear space is strictly convex.*

Proof. Let (X, d) be an LUR metric linear space and x, y with $x \neq y$ be such that $d(x, 0) = r = d(y, 0)$. Since $x \neq y, \epsilon = \frac{1}{2}d(x, y) > 0$ and $d(x, y) > \epsilon$. As (X, d) is LUR, there exists $\delta(\epsilon, x)$ such that $d\left(\frac{x+y}{2}, 0\right) \leq r - \delta(\epsilon, x) < r$. Therefore by Corollary 2.3, (X, d) is strictly convex. □

Remark 4.7. *An LUR space may not have the $2R$ property even in normed linear spaces. Smith [20] gave an example of an LUR Banach space that does not have the $2R$ property. Fan and Glicksberg [8] asked whether a space satisfying the $2R$ property is LUR, which was answered in negative by Polak and Sims [13].*

The sequential version of the LUR property can be stated as follows.

Proposition 4.8. *If a metric linear space (X, d) is LUR, then for any $x \in X$ with $d(x, 0) = r$ and for any sequence $\{y_n\}$ in $B[0, r]$ such that $d\left(\frac{x+y_n}{2}, 0\right) \to r$, we have $d(x, y_n) \to 0$. Converse holds in metric linear spaces with ball convexity.*

Proof. By taking $x_n = x$ for all n in the proof of Proposition 3.5, we obtain the desired result. □

Vlasov [24] called a normed linear space CLUR if $x \in S_X, \{x_n\} \subset S_X$ and $||\frac{x+x_n}{2}|| \to 1$ imply $\{x_n\}$ has a convergent subsequence. It is easy to see that an LUR normed linear space is CLUR but the following example shows that a CLUR normed linear space may not be LUR.

Example 4.9. [12, page 394] *Let $X = l^p, 1 < p < \infty$.*
For an element $x = (x_1, x_2, \ldots, x_n, \ldots) \in X$, let $x' = (0, x_2, \ldots, x_n, \ldots)$ and $x'' = (x_1, 0, x_2, \ldots, x_n, \ldots)$. If norm on X is defined as $||||x|||| = max\{||x'||_p, ||x''||_p\}$, then $(X, ||||.||||)$ is a CLUR space which is not strictly convex and hence cannot be LUR.

We now extend the notion of CLUR spaces to metric linear spaces as follows.

Definition 4.10. *A metric linear space (X, d) is called* CLUR *if for a positive real number r, whenever $d(x, 0) = r$, $d(x_n, 0) \leq r$ for all n, and $d\left(\frac{x+x_n}{2}, 0\right) \to r$, then $\{x_n\}$ has a subsequence converging to some $y \in B[0, r]$.*

As stated earlier, any LUR normed linear space is CLUR. The following result shows that this holds true for metric linear spaces also. The proof is immediate from Proposition 4.8.

Proposition 4.11. *Every LUR metric linear space is CLUR*

As mentioned earlier, a CLUR normed linear space may not be strictly convex or LUR but it is known (see [12, p. 394]) that a normed linear space which is both strictly convex and CLUR is LUR. We show that this holds true in case of metric linear spaces as well.

Proposition 4.12. *A metric linear space which is both strictly convex and CLUR is LUR.*

Proof. Let (X, d) be a metric linear space which is both strictly convex and CLUR. Assume to the contrary that (X, d) is not LUR. Then there exists $\epsilon > 0$ and $x \in X$ with $d(x, 0) = r$ for which no positive δ works. In particular, for each n, there is $x_n \in X$ satisfying $d(x_n, 0) \leq r, d(x, x_n) \geq \epsilon$ but $d\left(\frac{x+x_n}{2}, 0\right) > r - \frac{1}{n}$. As $x, x_n \in B[0, r]$ for all n and $x \neq x_n$, therefore, by strict convexity of (X, d) , we have $d\left(\frac{x+x_n}{2}, 0\right) < r$ for all n. Thus $d\left(\frac{x+x_n}{2}, 0\right) \to r$. Now as (X, d) is CLUR, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some $y \in B[0, r]$. Then, $d\left(\frac{x+x_{n_k}}{2}, 0\right) \to d\left(\frac{x+y}{2}, 0\right)$. Also, $d\left(\frac{x+x_{n_k}}{2}, 0\right) \to r$, thus by uniqueness of limit, we have $d\left(\frac{x+y}{2}, 0\right) = r$. But $x, y \in B[0, r]$; therefore, by strict convexity of (X, d), we have $x = y$. Thus $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ converging to x, which is not possible since $d(x, x_n) \geq \epsilon$ for all n. □

Anderson [4] called a normed linear space, midpoint locally uniformly convex or MLUR if whenever $\{x_n\}$ and $\{y_n\}$ are sequences in S_X and $\frac{1}{2}(x_n + y_n)$ converges to some member of S_X, then $||x_n - y_n|| \to 0$. It is known (see [11, p.473]) that every LUR normed linear space is MLUR and every MLUR normed linear space is strictly convex. We extend MLUR notion to metric linear spaces as follows.

Definition 4.13. *A metric linear space (X, d) is called MLUR, if whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $d(x_n, 0) \leq r, d(y_n, 0) \leq r$ and $d\left(\frac{x_n+y_n}{2}, z\right) \to 0$ for some $z \in X$ such that $d(z, 0) = r$, then $d(x_n, y_n) \to 0$, where $r > 0$ is any real number.*

Every LUR normed linear space is MLUR but as of now, we do not know whether this is true in case of metric linear spaces or not. We leave it as an open question.

Question 4.14. *Is every LUR metric linear space MLUR? If the answer is 'No', then whether every uniformly convex metric linear space is MLUR?*

We next see that every MLUR metric linear space with ball convexity is strictly convex.

Proposition 4.15. *An MLUR metric linear space with ball convexity is strictly convex.*

Proof. Let (X, d) be a metric linear space that has ball convexity and is MLUR. Suppose to the contrary that (X, d) is not strictly convex. Then there exists x, y $(x \neq y) \in X$, such that $d(x, 0) = r = d(y, 0)$ but $d\left(\frac{x+y}{2}, 0\right) \geq r$. As $x, y \in B[0, r]$ and $B[0, r]$ is convex, we have $d\left(\frac{x+y}{2}, 0\right) \leq r$. Thus, $d\left(\frac{x+y}{2}, 0\right) = r$. Taking $x_n = x$ for all n and $y_n = y$ for all n, we have $d(x_n, 0) = r = d(y_n, 0)$ and $\frac{1}{2}(x_n + y_n)$ converges to $z = \frac{1}{2}(x + y)$ but $d(x_n, y_n) = d(x, y) \neq 0$ for all n. Thus, $d(x_n, y_n)$ does not converge to 0, which contradicts the hypothesis that (X, d) is an MLUR space. □

The presence of ball convexity in normed linear spaces implies that MLUR normed linear spaces are strictly convex spaces, albeit we are unsure if this holds true for MLUR metric linear spaces as well, and we leave it as an open question.

Question 4.16. *Is every MLUR metric linear space strictly convex?*

5. Conclusion

One of the most natural themes in mathematical research is to refine the framework of the established results and see which ones hold up in situations that are a little more universal. Over the years, several different forms of convexity have been used in discussing existence and uniqueness concerns related to approximation problems in normed linear spaces. It is hoped that extending these notions of convexity to metric linear spaces would encourage further research in different branches of mathematics.

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References

[1] G.C. Ahuja, T.D.Narang and S. Trehan, Best Approximation on Convex Sets in Metric Linear Spaces, Math. Nachr., Vol. 78, No. 1, 125-130, 1977.
[2] N.I. Akhiezer and W. Fleming, Some Questions in The Theory of Moments, Amer. Math. Soc., Vol. 2, 1962.
[3] G. Albinus, Normaritge Metriken Auf Metrisierbaren Lokalkonvexen Topologischen Viktorräumen, Math. Nachr., Vol. 37, 177-196, 1968.
[4] K. W. Anderson, Midpoint Local Uniform Convexity, And Other Geometric Properties Of Banach Spaces, University of Illinois at Urbana-Champaign, 1960.
[5] J.A. Clarkson, Uniformly Convex Spaces, Trans. Amer. Math. Soc., Vol. 40, No. 3, 396-414, 1936.
[6] M.M. Day, Reflexive Banach spaces not isomorphic to uniformly convex spaces, Bull. Amer. Math. Soc., Vol. 47, No. 4, 313-317, 1941.
[7] K.Y. Fan and I. Glicksberg, Fully convex normed linear spaces, Proc. Natl. Acad. Sci. U. S. A, Vol. 41, No. 11, 947, 1955.
[8] K.Y. Fan and I. Glicksberg, Some geometric properties of the sphere in a normed linear space, Duke Math. J., Vol. 25, No. 4, 553-568, 1958.
[9] J. Fletcher and W.B. Moors, Chebyshev sets, J. Aust. Math. Soc., Vol. 98, No. 2, 161-231, 2015.
[10] A.R. Lovaglia, Locally uniformly convex Banach spaces, Trans. Amer. Math. Soc., Vol. 78, No. 1, 225-238, 1955.
[11] R.E. Megginson, An introduction to Banach space theory, Springer Science & Business Media, Vol. 183, 2012.
[12] H.N. Mhaskar and D.V. Pai, Fundamentals of approximation theory, CRC Press, 2000.

[13] T. Polak and B. Sims, A Banach space which is fully 2-rotund but not locally uniformly rotund, Canad. Math. Bull.,Vol. 26, No. 1, 118-120, 1983.
[14] S. Rolewicz, Metric linear spaces, D. Reidel Publishing Company, 1985.
[15] W. Rudin, Functional Analysis, Tata McGraw Hill, New Delhi, 1973.
[16] K.P.R. Sastry and S.V.R. Naidu, Convexity conditions in metric linear spaces, Mathematics seminar notes, Vol. 7, No. 2, 235-251, 1979.
[17] K.P.R. Sastry and S.V.R. Naidu, Uniform convexity and strict convexity in metric linear spaces, Math. Nachr., Vol. 104, No. 1, 331-347, 1981.
[18] K.P.R. Sastry, S.V.R. Naidu, I.H.N. Rao and K.P.R. Rao, Geometry of metric linear spaces with applications to fixed point theory, Tamkang J. Math., Vol. 18, 1-12, 1987.
[19] I. Singer, The Theory of Best Approximation and Functional Analysis, SIAM, 1974.
[20] M.A. Smith, A reflexive Banach space that is LUR and not 2R, Canad. Math. Bull., Vol. 21, No. 2, 251-252, 1978.
[21] M.A. Smith, Some examples concerning rotundity in Banach spaces, Math. Ann., Vol. 233, No. 2, 155-161, 1978.
[22] V. Šmulian, On some geometrical properties of the unit sphere in the space of the type (B), Mat. Sb., Vol. 48, No. 1, 77-94, 1939.
[23] A.I. Vasil'ev, Chebyshev sets and strong convexity of metric linear spaces, Math. Notes, Vol. 25, No. 5, 335-340, 1979.
[24] L.P. Vlasov, Chebyshev sets and approximately convex sets, Math. Notes, Vol. 2, No. 2, 600-605, 1967.

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