Electronic Journal of Mathematical Analysis and Applications
Vol. 13(1) Jan. 2025, No.6.
ISSN: 2090-729X (online)
ISSN: 3009-6731(print)
http://ejmaa.journals.ekb.eg/

FUZZY EFFECT ALGEBRAS AND DECOMPOSITION THEOREMS

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ABSTRACT. This paper delves the concept of charges on fuzzy effect algebras (FEAs). We explore the fundamental properties of charges on FEAs, providing a comprehensive analysis with its suitable examples. The space of all bounded charges on FEAs has also been explored. We prove Jordan decomposition theorem for charges on FEAs. Finally, we also prove Hahn decomposition theorem for charges defined on FEAs.

1. INTRODUCTION

Effect algebras intoduced by Bennett and Foulis [4] are an innovative class of mathematical structures that contains the essence of quantum mechanics and offers a framework for modeling quantum observables and their partial order relations. FEAs are a fast-growing topic that emerged from the combination of effect algebras with fuzzy set theory. FEAs were motivated by the need for greater flexibility in the applicability of effect algebras to scenarios with inherent uncertainty and imprecision. This combination not only extends the reach of effect algebras but also overcomes the difficulties presented by indefinite real-world systems.

Decomposition theorems demonstrate themselves to be effective instruments for analysing the structure and behaviour of complex mathematical objects. These theorems offer an approach of dividing complex structures into simple parts, highlight hidden patterns, and provide significant new insights of the underlying mechanisms directing an individual mathematical discipline. The French mathematician C. Jordan introduced the Jordan decomposition theorem which states that the signed measure μ admits a unique decomposition into a difference $\mu = \mu^+ - \mu^-$ of two positive measures, at least one of which is finite, and such that for any Hahn decomposition $\Omega^+ \cup \Omega^-$ and measurable set A, if $A \subseteq \Omega^-$ then $\mu^+(A) = 0$ and if

2020 *Mathematics Subject Classification.* 03E70, 28A12, 46L45, 06G12.
Key words and phrases. charges, effect algebras, decomposition theorems.
Submitted July 05, 2024. Revised Aug. 13, 2024.

$A \subseteq \Omega^+$ then $\mu^-(A) = 0$; for different type of decomposition theorems on different spaces, see ([2], [8-9], [11], [13-15] and [20]). This theorem finds applications in diverse fields such as quantum mechanics, control theory, and numerical analysis. An important finding which provides an approach to separate the space of a signed measure into positive and negative sections is the Hahn decomposition Theorem [9]. This theorem, developed by the German mathematician Hahn, is important for understanding the theory of integration and probability and also important for understanding the structure of measures. When a measure includes both positive and negative values, the Hahn Decomposition theorem helps us to determine the positive and negative components in a structured and efficient way. It also offers a simple and comprehensible division of the measure's positive and negative contributions.

In the present paper, we have introduced the concept of charges on FEAs. We prove Jordan and Hahn decomposition theorems in the context of FEAs as our main contribution.

The manuscript is organized as follows: Section 2 contains basics of effect algebras and FEAs. In section 3, charges are introduced on FEAs and a few examples are also given. Section 4 introduces and investigates properties of the space of all bounded charges on FEAs. In section 5, we prove Jordan decomposition theorem and in section 6, Hahn decomposition theorem is proved on FEAs. Section 7 is the conclusion of the paper.

2. Basic concepts

In this section, we introduce some basic concepts, definitions and results from FEAs.

2.1. **Fuzzy effect algebras.**

An effect algebra (introduced by D. J. Foulis and M. K. Bennett [4]) is a system $(E, \oplus, 0_E, 1_E)$, where E is a set, 0_E and 1_E are special elements of E, called the zero and the unit, and $\oplus$ is a partially defined binary operation on E such that for $f, g, h \in E$ the following axioms are assumed:

(i) If $f \oplus g$ is defined, then $g \oplus f$ is defined and $f \oplus g = g \oplus f$ (Commutative law).

(ii) If $g \oplus h$ is defined and $f \oplus (g \oplus h)$ is defined, then $f \oplus g$ and $(f \oplus g) \oplus h$ are defined and $f \oplus (g \oplus h) = (f \oplus g) \oplus h$ (Associative law).

(iii) For every $f \in E$ there exists a unique elements $g \in E$ such that $f \oplus g$ is defined and $f \oplus g = 1_E$ (Orthosupplement law).

(iv) If $f \oplus 1_E$ is defined, then $f = 0_E$ (Zero-one law).

In every effect algebra E, a dual operation $\ominus$ to $\oplus$ can be defined as follows: $f \ominus h$ exists and equals $g \iff g \oplus h$ exists and equals f. Two elements $f, g \in E$ are said to be *orthogonal* and written as $f \perp g$, if $f \oplus g$ exists. If $f \oplus g = 1$, then g is *orthocomplement* of f and write $g = f^\perp$. Obviously $1^\perp = 0$, $(f^\perp)^\perp = f$, $f \perp 0$ and $f \oplus 0 = f$, for all $f \in E$. Also for $f, g \in L$, define $f \leq g$ if there exists $h \in E$ such that $f \perp h$ and $f \oplus h = g$. Observe that $\leq$ is a partial ordering on E and $0 \leq f \leq 1$; $f \leq g \Leftrightarrow g^\perp \leq f^\perp$ and $f \leq g^\perp \Leftrightarrow f \perp g$ for $f, g \in E$. If $f \leq g$, the element $h \in E$ such that $h \perp f$ and $f \oplus h = g$ is unique, and satisfies the condition $h = (f \oplus g^\perp)^\perp$. We write it as $h = g \ominus f$.

For $f_1, \ldots, f_n \in E$, define $f_1 \oplus \ldots \oplus f_n = (f_1 \oplus \ldots \oplus f_{n-1}) \oplus f_n$ inductively, provided that the right hand side exists (it is independent on permutation of the elements). A finite subset $\{f_1, \ldots, f_n\}$ of E is said to be *orthogonal* if $f_1 \oplus \ldots \oplus f_n$

exists. A sequence $\{f_n\}$ in E is called orthogonal if, for every n, $\bigoplus_{i\leqslant n} f_i$ exists. If, moreover $\sup_n \bigoplus_{i\leqslant n} f_i$ exists, the *sum* $\bigoplus_{n\in\mathcal{N}} f_n$ of an orthogonal sequence $\{f_n\}$ in E is defined as $\sup_n \bigoplus_{i\leqslant n} a_i$. An effect algebra L is called a *σ-complete effect algebra* if every orthogonal sequence in E has its sum. If $(E,\leq)$ is a lattice, we say that effect algebra E is a *lattice effect algebra*. For more literature on effect algebras, see ([2-5], [7], [10], [12] and [16-19]).

An effect algebra E can be mapped to another effect algebra H through a map ϕ that preserves the structure of effect algebras, a property referred to as EA-structure. Such a map is known as an EA-homomorphism. Consider the closed unit interval I with its typical EA-structure, where the operation $f\oplus g$ is defined whenever $f+g\leq 1$ and in such cases, $f\oplus g$ is simply $f+g$. Hence, the system $(I,+,0,1)$ forms an effect algebra. Similarly, if E is a set and I^E represents the collection of all functions from E into I, then I^E can be treated as an effect algebra with a partial operation $\oplus_{\mathcal{E}}$ defined point-wise. Specifically, $f\oplus_{\mathcal{E}} g$ exists if and only if $f(x)+g(x)\leq 1$ for all $x\in E$ and it is given by $(f\oplus_{\mathcal{E}} g)(x)=f(x)+g(x)$ for each $x\in E$. A subset $\mathcal{E}\subset I^E$ that includes the constant functions $0_{\mathcal{E}}$ and $1_{\mathcal{E}}$ and is closed under the inherited partial operation $\oplus_{\mathcal{E}}$ constitutes an effect algebra known as the *effect algebra of fuzzy sets* or simply *fuzzy effect algebras*, see [17]). In general, $\mathcal{E}$ is assumed to denote the system $(\mathcal{E},\oplus_{\mathcal{E}},0_E,1_E)$ throughout the manuscript. We denote by $\mathcal{N}$ the set of all positive integers and by $\mathcal{R}$ the set of real numbers. Hence we can consider the following effect algebra as an example of FEAs:

Example 2.1. *Let $X\neq\phi$ be a set and let $\mathcal{E}\subseteq[0,1]^X$. We call $\mathcal{E}$ a fuzzy set system on X if*

(i) $0,1\in\mathcal{E}$,
(ii) $f\in\mathcal{E}$ then $1-f\in\mathcal{E}$,
(iii) $f,g\in\mathcal{E}$ with $f+g\leqslant 1$ then $f+g\in\mathcal{E}$.
The system $\mathcal{E}$ represents a fuzzy effect algebra when $f\oplus_{\mathcal{E}} g=f+g$ for $f+g\leqslant 1$.

3. Charges on fuzzy effect algebras

In this section we shall introduce the concept of charges on FEAs $\mathcal{E}$. They are usually said as finitely additive measures in the literature, which arise quite naturally in many areas of science and engineering and have been widely used by mathematicians and statisticians over the years. The main idea of the paper has been taken from Rao and Rao [6].

Definition 3.1. *A map $m:\mathcal{E}\to[-\infty,\infty]$ is said to be charges on $\mathcal{E}$ if*
(i) $m(0)=0$;
(ii) $m(f\oplus_{\mathcal{E}} g)=m(f)+m(g)$ whenever $f\perp g, f,g\in\mathcal{E}$.

If m is a charge on $\mathcal{E}$, it can not take both the value $+\infty$ and $-\infty$. For if $f,g\in\mathcal{E}$, such that $m(f)=\infty$ and $m(g)=-\infty$. Then $m(1)=m(f\oplus_{\mathcal{E}} f^{\perp})=\infty=m(g\oplus_{\mathcal{E}} g^{\perp})=-\infty$, which is a contradiction. Following is a example of a charge on $\mathcal{E}$.

Example 3.1. *Let us consider the lattice effect algebra $\mathcal{E}=\{0,a,b,c,1\}$ where we define : $a\oplus b=b\oplus a=c$, $b\oplus c=c\oplus b=a\oplus a=1$ and let $x\oplus 0=0\oplus x$ for all $x\in E$. Define the function $m:\mathcal{E}\to[0,1]$ as follows: $m(x)=0$ if $x\in\{c,b\}$, $m(x)=1/2$ if $x\in\{0,a\}$, and $m_1(1)=1$. Then m is a charge on $\mathcal{E}$.*

Definition 3.2. *Let m be a charges on a $\sigma-$complete FEA $\mathcal{E}$. The charge m is said to be s-bounded (called exhaustive in [14]) if for every sequences $\{f_n\}n \geq 1$ of mutually orthogonal elements in $\mathcal{E}$, $\lim n \to \infty m(f_n) = 0$.*

Note that if m is a real charge for every $f \in \mathcal{E}$ (that is, m takes values in $(-\infty, \infty)$) then m is s-bounded on $\mathcal{E}$ if and only if $\sup\{|m(f)|; f \in L\} < \infty$. Consider the following example from [14]:

Example 3.2. *Consider the effect algebra $\mathcal{E} = \{0, 1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{p}, ...\}$, where we define: for each $\frac{1}{p}$, $0 \oplus \frac{1}{p} = \frac{1}{p}$, $\frac{1}{p} \oplus \frac{1}{p} = 1$, $0 \oplus 1 = 1$ and if $p \neq q$, $\frac{1}{p} \oplus \frac{1}{q}$ cannot be defined. Consider the function $m : \mathcal{E} \to [0, 1]$ defined by $m(x) = 1$ if $x = 1$, and otherwise $m(x) = 0$. Then m is s-bounded on $\mathcal{E}$.*

Lemma 3.1. *Let m be a real charges on a $\sigma-$complete FEA $\mathcal{E}$. If m is unbounded, there exist $f_1, f_2 \in \mathcal{E}$ satisfying*
(i) $f_1 \perp f_2$;
(ii) $|m(f_1)| \geq 1$ and $|m(f_2)| \geq 1$;
(iii) m is unbounded either on f_1 or on f_2 in $\mathcal{E}$ i.e. $\sup\{|m(g)| : g \leq f_1, g \in \mathcal{E}\} = \infty$ or $\sup\{|m(h)| : h \leq f_2, h \in L\} = \infty$.

Proof. Since is m is s-bounded, there exists $g_1 \in \mathcal{E}$ such that $|m(g_1)| \geq |m(1)| + 1$. Then $|m(g_1^\perp)| = |m(1 \ominus_{\mathcal{E}} g)| = |m(1) - m(g)| \geq ||m(1)| - |m(g_1)|| \geq |m(g_1)| - |m(1)| \geq 1$. It is clear that m is unbounded either on g_1 or on $g_1^\perp$. Let us take $f_1 = g_1$ and $f_2 = g_1^\perp$. Clearly $|m(f_1)| \geq 1, |m(f_2)| \geq 1$. □

Theorem 3.1. *Let m be a real but unbounded charge on a $\sigma-$complete fuzzy effect algebra $\mathcal{E}$. Then there exist a sequence $\{f_n\}, n \geq 1$ of mutually orthogonal element from $\mathcal{E}$ such that $|m(f_n)| \geq 1$ for every $n \geq 1$.*

Proof. By Lemma 3.1, there exists $f_1, g_1 \in \mathcal{E}$ such that $f_1 \perp g_1$ with $|m(f_1)| \geq 1, |m(g_1)| \geq 1$ and m is unbounded on g_1. Applying Lemma 3.1, again on g_1, there exists $f_2, g_2 \in \mathcal{E}$ such that $f_2 \leq g_1, g_2 \leq g_1$ and $f_2 \perp g_2$ with $|m(f_2)| \geq 1, |m(g_2)| \geq 1$ and m is unbounded on g_2. Repeating the process, we obtain a sequence $\{f_n\}_{n \geq 1}$ of mutually orthogonal elements from $\mathcal{E}$ such that $|m(f_n)| \geq 1$, for every n. □

Theorem 3.2. *Let m be a real charge on $\sigma-$complete FEA $\mathcal{E}$. Then m is s-bounded on $\mathcal{E}$ if and only if m is bounded on $\mathcal{E}$.*

Proof. Let m be bounded on $\mathcal{E}$. Let $k = \sup\{|m(g)| : g \in \mathcal{E}\}$, Then k is finite. Let $\{f_n\}_{n \geq 1}$ be any sequence of mutually orthogonal elements in $\mathcal{E}$. Let $m \geq 1$ be fixed. Let $S_1 = \{1 \leq i \leq m; m(f_i) \geq 0\}$ and $S_2 = \{1 \leq i \leq m; m(f_i) \leq 0\}$. Then

$$\sum_{i=1}^{m} |m(f_i)| = \sum_{i \in S_1} m(f_i) - \sum_{j \in S_2} m(f_j)$$

$$= m(\oplus_{i \in S_1 \mathcal{E}} f_i) - m(\oplus j \in S_{2\mathcal{E}} f_j) \leq 2k.$$

Hence $\sum_{i \geq 1} |m(f_i)| \leq 2k$. It gives that $\lim_{n \to \infty} m(f_n) = 0$. So m is s-bounded on $\mathcal{E}$. □

Conversely, let m is s-bounded on $\mathcal{E}$ but not bounded on $\mathcal{E}$. By Theorem 3.1, there exist a sequence $\{f_n\}n \geq 1$ of mutually orthogonal elements in $\mathcal{E}$ such that $|m(f_n)| \geq 1$ for every $n \geq 1$. Thus given that $\lim_{n \to \infty} m(f_n) \neq 0$. Contradicting the fact that m is s-bounded on $\mathcal{E}$.

4. The space $BA(\mathcal{E})$ of all bounded charges

The space of all bounded charges defined on an effect algebras $\mathcal{E}$ is denoted by $BA(\mathcal{E})$. A natural ordering " $\leq$ " on $BA(\mathcal{E})$ can be defined as: for $m, n \in BA(\mathcal{E}), m \leq n$ if $m(f) \leq n(f); \forall f \in \mathcal{E}$. It can be proved that the relation " $\leq$ " is reflexive, anti-symmetric and transitive i.e " $\leq$ " is a partial order on $\mathcal{E}$. Throughout this section we shall assume that $\mathcal{E}$ be a lattice FEA satisfying the conditions:

(A1) $g \wedge (f_1 \oplus f_2) = (g \wedge f_1) \oplus (g \wedge f_2)$ for $f_1, f_2, f_1 \perp f_2, g \in \mathcal{E}$;

(A2) If $f_1, f_2, f_1 \perp f_2, g_1, g_2, g_1 \leq f_1, g_2 \leq f_2, g \in \mathcal{E}$, then $(f_1 \oplus f_2) \ominus (g_1 \oplus g_2) = (f_1 \ominus g_1) \oplus (f_2 \ominus g_2)$.

Theorem 4.3. *Let $\mathcal{E}$ be a lattice FEA satisfying the conditions (A1) and (A2). Then following statements are true:*

(i) If $m_1, m_2 \in BA(\mathcal{E})$, then $cm_1 + dm_2 \in BA(\mathcal{E})$, where $c, d \in \mathcal{R}$;

(ii) Let $m_1, m_2 \in BA(\mathcal{E})$. Define λ on $\mathcal{E}$ by $\lambda(g) = \sup\{m_1(f) + m_2(g \ominus_{\mathcal{E}} f); f \leq g, f, g \in \mathcal{E}\}$. Then $\lambda \in BA(\mathcal{E})$;

(iii) Let $m_1, m_2 \in BA(\mathcal{E})$. Then $m_1 \vee m_2$ exists and $m_1 \vee m_2 = \lambda$ on $BA(\mathcal{E})$;

(iv) Let $m_1, m_2 \in BA(\mathcal{E})$. Define τ on L as $\tau(g) = \inf\{m_1(f) + m_2(g \ominus_{\mathcal{E}} f); f \leq g, f, g \in \mathcal{E}\}$.Then $\tau \in BA(\mathcal{E})$;

(v) Let $m_1, m_2 \in BA(\mathcal{E})$. Then $m_1 \wedge m_2$ exists and $m_1 \wedge m_2 = \tau$ on $BA(\mathcal{E})$.

Proof. The proof of (i) is obvious.

(ii) Clearly $\lambda(0) = 0$. Since m_1 and m_2 are bounded on $\mathcal{E}$, λ is bounded function on $\mathcal{E}$. Let $f_1, f_2 \in \mathcal{E}$ such that $f_1 \perp f_2$. Let $g \in BA(\mathcal{E})$ such that $g \leq f_1 \oplus_{\mathcal{E}} f_2$ and let $g_1 \leq f_1$ (we may choose $g_1 = g \wedge f_1, g_1 \in \mathcal{E}$) and $g_2 \leq f_2$ (we may choose $g_2 = g \wedge f_2, g_2 \in \mathcal{E}$). Then

$$m_1(g) + m_2[(f_1 \oplus_{\mathcal{E}} f_2) \ominus_{\mathcal{E}} g] = m_1(g_1 \oplus_{\mathcal{E}} g_2) + m_2[(f_1 \ominus_{\mathcal{E}} g_1) \oplus_{\mathcal{E}} (f_2 \ominus_{\mathcal{E}} g_2)]$$

$$= m_1(g_1) + m_2(f_1 \ominus_{\mathcal{E}} g_1) + m_1(g_2) + m_2(f_2 \ominus_{\mathcal{E}} g_2) \leq \lambda(f_1) + \lambda(f_2)$$

(as $g = g_1 \oplus g_2, g_1, g_2 \in \mathcal{E}$). Taking supremum over all $g \leq f_1 \oplus_{\mathcal{E}} f_2$ in $\mathcal{E}$, we get $\lambda(f_1 \oplus_{\mathcal{E}} f_2) \leq \lambda(f_1) + \lambda(f_2)$. Also,

$$[m_1(g_1) + m_2(f_1 \ominus_{\mathcal{E}} g_1)] + [m_1(g_2) + m_2(f_2 \ominus_{\mathcal{E}} g_2)]$$

$$= m_1(g_1 \oplus_{\mathcal{E}} g_2) + m_2[(f_1 \oplus_{\mathcal{E}} f_2) \ominus_{\mathcal{E}} (g_1 \oplus_{\mathcal{E}} g_2)] \leq \lambda(f_1 \oplus_{\mathcal{E}} f_2).$$

Now taking supremum over all $g_1 \leq f_1, g \in \mathcal{E}$ and $g_2 \leq f_2, g_2 \in \mathcal{E}$, we get

$$\lambda(f_1) + \lambda(f_2) \leq \lambda(f_1 \oplus_{\mathcal{E}} f_2).$$

Thus, we have $\lambda(f_1 \oplus_{\mathcal{E}} f_2) = \lambda(f_1) + \lambda(f_2)$. Hence $\lambda \in BA(\mathcal{E})$.

(iii) It is clear that $m_1 \leq \lambda, m_2 \leq \lambda$. Let $m \in BA(\mathcal{E})$ be such that $m_1 \leq m, m_2 \leq m$. Then for any $g \in \mathcal{E}$,

$$\lambda(g) = \sup\{m_1(f) + m_2(g \ominus_{\mathcal{E}} f) : f \leq g, f \in \mathcal{E}\}$$

$$\leq \sup\{m(f) + m(g \ominus_{\mathcal{E}} f) : f \leq g, f \in \mathcal{E}\} = m(g).$$

Hence $\lambda \leq m$. Consequently $m_1 \vee m_2$ exists in $\mathcal{E}$ and $m_1 \vee m_2 = \lambda$.

(iv) Proof is similar as (ii).

(v) Proof is similar as (iii). □

5. Jordan decomposition theorem for charges on fuzzy effect algebras

Definition 5.3. *Let $\mathcal{E}$ be a lattice FEA. Let m_1 and m_2 be two charges on $\mathcal{E}$. Define λ and τ on $\mathcal{E}$ by*
(i) $\lambda(g) = \sup\{m_1(f) + m_2(g \ominus_{\mathcal{E}} f); f \leq g, f, g \in \mathcal{E}\}$;
(ii) $\tau(g) = \inf\{m_1(f) + m_2(g \ominus_{\mathcal{E}} f); f \leq g, f, g \in \mathcal{E}\}$.

Recall from the Theorem 4.3 that if either both m_1 and m_2 do not take $+\infty$ or both m_1 and m_2 do not take $-\infty$, then $\lambda = m_1 \vee m_2$ and $\tau = m_1 \wedge m_2$ are charges on $\mathcal{E}$. This definition is consistent with the definitions used in (ii) and (v) of the Theorem 4.3.

Theorem 5.4. *(Jordan decomposition theorem). Let $\mathcal{E}$ be a lattice FEA satisfying the conditions (A1) and (A2) and let m be a charge on $\mathcal{E}$. Define m^+ and m^- by*

$$m^+(f) = \sup\{m(g) : g \leq f, g \in \mathcal{E}\}, f \in \mathcal{E}$$

and

$$m^-(f) = -\inf\{m(g); g \leq f, g \in \mathcal{E}\}, f \in \mathcal{E}.$$

Here m^+ and m^- are the positive and negative variations of m on $\mathcal{E}$, respectively. We have the following properties:
(i) *m^+ and m^- are positive charges on $\mathcal{E}$;*
(ii) *If $m \neq +\infty$, then $m^+ - m = m^-$;*
(iii) *If $m \neq -\infty$, then $m + m^- = m^+$;*
(iv) *If $m \neq +\infty$ and $m_1 - m = m_2$ for some positive charges m_1 and m_2 on $\mathcal{E}$, then $m_1 \geq m^+, m_2 \geq m^-$;*
(v) *If $m \neq -\infty$ and $m + \lambda_1 = \lambda_2$ for some positive charges λ_1 and λ_2 on $\mathcal{E}$ then $\lambda_1 \geq m^-$ and $\lambda_2 \geq m^+$;*
(vi) *$m = m^+ - m^-$ if and only if m is either bounded below or bounded above;*
(vii) *$m^+ \wedge m^- = 0$ if and only if m is either bounded below or bounded above;*
($viii$) *If m_1 and m_2 are positive charges on $\mathcal{E}$ such that $m = m_1 - m_2$ and $m_1 \wedge m_2 = 0$, then $m_1 = m^+$ and $m_2 = m^-$;*
(ix) *If m is a real charge on $\mathcal{E}$ then $m = m^+ - m^-$ holds if and only if m is bounded. In this case, both m^+ and m^- are bounded.*

Proof. (i) From Definition 5.3, observe that: $m^+ = m \vee 0$ and $m^- = (-m) \vee 0$ and hence m^+ and m^- are charges on $\mathcal{E}$.

(ii) Let $g \in \mathcal{E}$. Suppose $m(g) = -\infty$. Then from definition of m^-, $m^-(g) = \infty$. Therefore $m^+(g) - m(g) = \infty = m^-(g)$. Let us suppose that $m(g) > -\infty$. By the assumption, $-\infty < m(g) < \infty$. Consequently, $-\infty < m(f) < \infty$ for any $f \in \mathcal{E}$ such that $f \leq g$. By the Theorem 4.3 we have,

$$\begin{aligned} m^+(g) - m(g) &= \sup\{m(f) : f \leq g, f \in \mathcal{E}\} - m(g), \\ &= \sup\{m(f) - m(g) : f \leq g, f \in \mathcal{E}\} \\ &= \sup\{-m(g \ominus_{\mathcal{E}} f) : f \leq g, f \in \mathcal{E}\} \\ &= -\inf\{m(h) : h \leq g, h \in \mathcal{E}\} = m^-(g). \end{aligned}$$

(iii) Proof is similar as (ii).

(iv) Since $m_1 - m = m_2$ and m_1 and m_2 are positive charges on $\mathcal{E}$, we have $m_1 \geq m$, so for any $g \in \mathcal{E}$, $m^+(g) = \sup\{m(f); f \leq g, f \in L\} \leq m_1(g)$. Hence $m^+ \leq m_1$. For second part, note that $m^+ - m \leq m_1 - m = m_2$. By (ii) $m^+ - m = m^-$.

(v) Proof is similar as (iv).

(vi) Let m be bounded above. By (ii), $m^+ - m = m^-$, and m^+ is a bounded charge on $\mathcal{E}$, hence $-m = m^- - m^+$ or $m = m+ - m^-$. Logic is similar for the case m is bounded below on $\mathcal{E}$. Conversely, if $m = m^+ - m^-$, then either m^+ is bounded or m^- is bounded. In previous case m is bounded above and later it is bounded below.

(vii) Let m be bounded below. Then m^- is positive bounded charge on $\mathcal{E}$. So, for any $f \in \mathcal{E}$,

$$(m^+ \wedge m^-)(g) = \inf\{m^+(f) + m^-(g \ominus_{\mathcal{E}} f) : f \leq g, f \in \mathcal{E}\}$$

$$= \inf\{m(f) + m^-(f) + m^-(g \ominus_{\mathcal{E}} f) : f \leq g, f \in \mathcal{E}\};$$

$$= \inf\{m(f) + m^-(g) : f \leq g, f \in \mathcal{E}\};$$

$$= \inf\{m(f) : f \leq g, f \in \mathcal{E}\} + m^-(g) = -m^-(g) + m^-(g) = 0$$

(since m^- is bounded).

Similarly, we can prove that if m is bounded above the $m^+ \wedge m^- = 0$.

Conversely, let us suppose that m is neither bounded below nor bounded above. Without loss of generality, we can assume that m does not take value $-\infty$.

Note that $m^-(1) = \infty$, By (iii), $m + m^- = m^+$, so

$$(m^+ \wedge m^-)(1) = [(m + m^-) \wedge m^-](1)$$

$$= \inf\{(m + m^-)(g) + m^-(g^\perp) : g \leq 1, g \in \mathcal{E}\}$$

$$= \inf\{m(g) + m^-(1) : g \in \mathcal{E}\} = \infty \Rightarrow m^+ \wedge m^- \neq 0.$$

This completes the proof.

($viii$) If $m = m_1 - m_2$, where m_1 and m_2 are positive charges, then m is either bounded below or bounded above (from (vi)). Let us assume m is bounded above, which yield that m is a bounded charge. Since $m = m_1 - m_2 = m^+ - m^-$, we have $m_1 - m^+ - m_2 = -m^- \leq 0$. Therefore, $m_1 - m^+ \leq m_2$. By (iv), $m_1 \geq m^+$, so $0 \leq m_1 - m^+ \leq m_2$. Since $m_1 \wedge m_2 = 0$ and $0 \leq m_1 - m^+ \leq m_1$, it gives that $0 \leq m_1 - m^+ \leq m_1 \wedge m_2 = 0$. Thus $m_1 = m^+$. Similarly we can prove that $m_2 = m^-$.

(ix) If m is a real charge and $m = m_1 - m_2$, where m_1 and m_2 are positive charges, then m_1 and m_2 are bounded. Hence m is bounded. If m is bounded, definitely m can be written as a difference of two positive charges on $\mathcal{E}$. □

6. Hahn decomposition theorem for charges on fuzzy effect algebras

Definition 6.4. *Let $\mathcal{E}$ be a FEA and m is a charge on $\mathcal{E}$. Let $\epsilon > 0$. Then a partition $\{f, f^\perp\}$ of 1 is said to be ϵ-Hahn decomposition of m if the following conditions are satisfied:*

(i) $g \in L, g \leq f \Rightarrow m(g) \leq \epsilon$,

(ii) $h \in L, h \leq f^\perp \Rightarrow m(h) \geq -\epsilon$.

Theorem 6.5. *(Hahn decomposition theorem). Let $\mathcal{E}$ be a lattice FEA satisfying the conditions (A1) and (A2) and let m be a charge on a FEA $\mathcal{E}$, which is either bounded below or bounded above. Then for any $\epsilon > 0$, there exists a ϵ-Hahn decomposition of m. If m is neither bounded below nor bounded above, there exists $\epsilon > 0$ for which there is no $\epsilon-$Hahn decomposition of m.*

Proof. Let m is bounded below. Let $d = \inf\{m(g) : g \in \mathcal{E}\}$. Then d is a finite number. Let $\epsilon > 0$, we can find $f \in L$ such that $d \leq m(f) \leq d + \epsilon$, which gives that $-\infty < m(f) < \infty$ and for any $g \in L, g \leq f$, we have $-\infty < m(g) < \infty$. So $d \leq m(f \ominus_{\mathcal{E}} g) = m(f) - m(g) \leq d + \epsilon - m(g)$, giving $m(g) \leq \epsilon$. Now let $h \in L, h \leq f^{\perp}$. Then $h \perp f$. Also, $d \leq m(h \oplus_{\mathcal{E}} f) = m(h) + m(f) \leq m(h) + d + \epsilon$. Hence $m(h) \geq -\epsilon$. This prove that the first part of the theorem.

The statement, that is admits $\epsilon-$Hahn decomposition for every $\epsilon > 0$ is equivalent to the statement that $m^+ \wedge m^- = 0$ (by (vi) and (vii) of the Theorem 5.4) which is equivalent to the statement that m is either bounded below or bounded above by the Theorem 5.4. □

7. Conclusion

The present paper deals with the notion of charges (which also have been called finitely additive measures in literature, see [6]) on FEAs and its application in obtaining the decomposition theorems on FEAs. Jordan decomposition theorem and Hahn decomposition theorem for charges on FEAs have also been achieved. The theory of fuzzy sets [21] and intuitionistic fuzzy sets [1], which have applications in many different domains including pattern detection and decision making, might gain attention significantly from this introduction of charges and decomposition theorems on FEAs.

8. Conflict of interest

No conflict of interest has been declared by the authors.

9. Ethical statement

This article does not contain any studies with human participants or animals performed by the authors.

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