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SOME REMARKS ON $F-g-$CONTRACTIONS IN METRIC SPACES

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Abstract. The aim of this paper is to significantly improved and supplement the recently established results from the papers (R. Batra, S. Vashistha, Coincidence Point Theorem for a New Type of Contraction on Metric Spaces, Int. Journal of Math. Analysis, Vol. 8, 2014, no. 27, 1315 - 1320 and D. Wardowski, N. Van Dung, Fixed points of $F-$weak contractions on complete metric spaces, Demonstratio Mathematica Vol. XLVII No 1. 2014, 146-155, about $F-g-$contractions and $F-$weak contractions. In the entire paper, for the Wardowski function F, we assume only its strict increasing on $(0,+\infty)$ i.e., the property F1. In both papers, the authors assume all three properties F1, F2 and F3 of the mapping F.

1. Introduction and preliminaries

The accelerated development of nonlinear analysis and with it the fixed point theory in both topological and ordinary metric spaces occurred at the end of the nineteenth and the beginning of the twentieth century. First, we received the works of Picard on successive approximations and then the works of H. Poincaré on topological spaces. Somewhere around 1912, Brouwer's work [6] appeared on the existence of a fixed point of every continuous mapping defined on the unit circle of the Cartesian plane. It is one of the first and most important works in topological fixed point theory. Later, other important works were arranged. After that, at the same time, the metric theory of the fixed point began to develop, which found application in most of the natural sciences, engineering, economics, medicine, and statistics. It is further combined with fractional calculus, the theory of differential and integral equations. In 1922, the Polish mathematician Stefan Banach [2] proved his famous theorem about the uniqueness of the fixed point of every contraction on

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the complete metric space. Let's recall its formulation: Let T be the mapping of the complete metric space (X, d) into itself and let there exist a λ from $[0,1)$ such that for every x, y from X, $d(Tx, Ty) \leq \lambda \cdot d(x, y)$. Then there exists a unique point u from X such that $Tu = u$.What's more, for every point x in X, the corresponding Picard sequence $T^n x$ converges to that point u.

Note that the work of Caccioppoli [7] from 1930 is also worth mentioning as one of the first works that generalizes the famous Banach result from 1922. After that, some authors mention it together with Banach's work in their works.

In the last little more than a hundred years, many research mathematicians have tried to generalize the mentioned Banach theorem. A large number of beautiful papers were written that continue to motivate many mathematicians. One of the significant results that generalizes Banach's famous theorem is also the result of the Polish mathematician D. Wardowski [11]. After that result, more new ones were created, which now generalize Wardowski.

In his work from 2012, D. Wardowski first introduced the class $\mathcal{F}$ of mappings from $(0, +\infty)$ to $\mathbb{R}$ as follows: Let F maps $(0, +\infty)$ to $\mathbb{R}$. We say that F belongs to the class of functions $\mathcal{F}$ if the following three properties are fulfilled for it:

F1) F is a strictly increasing function, that is., if $0 < \alpha < \beta < +\infty$ then $F(\alpha) < F(\beta)$;

F2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to+\infty} \alpha_n = 0$ if and only if $\lim_{n\to+\infty} F(\alpha_n) = -\infty$;

F3) There exist $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We now list several interesting examples of functions F that map $(0, +\infty)$ to $\mathbb{R}$ and satisfy all three properties **F1)**, **F2)** and **F3)**.

1.1. **Examples. 1.** $F(r) = \ln r$,

2. $F(r) = r + \ln r$,

3. $F(r) = -\frac{1}{\sqrt{r}}$,

4. $F(r) = \ln\left(r^2 + r\right)$,

etc,....

In addition to the mentioned properties **F1), F2)** and **F3)**, some authors replaced some of the mentioned properties with some other new property and considered the corresponding contractive condition as D. Wardowski in [11]. For example, F is a continuous mapping.

For such details see paper [8].

For more others details on $F-$contractions also see [4], [10] and [12].

Now we state a question for which we do not know the answer yet:

Is there such an F that satisfies F1) but does not satisfy both F2) and F3)?

When considering a pair of mappings (T, g) from the non-empty set X to itself, we state the following terms:

1) T and g are commutative mappings if $Tgx = gTx$ for every x in X;

2) If $Tx = gx$ for some one from X, then the point x is called a coincidence point, while the point $y = Tx = gx$ is called a point of coincidence.

3) A pair (T, g) is called weakly compatible if T and g commute at the coincidence point, i.e, if $Tgx = gTx$ whenever $Tx = gx$;

4) The mappings T and g of the pair (T, g) in the metric space (X, d) are compatible if $d(Tgx_n, gTx_n)$ tends to zero when n tends to $+\infty$ where x_n is a convergent sequence in the metric space (X, d).

All these notions are closely related to the notion of a common fixed point of the mapping T and g, especially in relation from the uniqueness of that common fixed point. The following rersult is from [1]:

For other interesting details see [5] and [9].

Proposition 1.1. [1] *Let T and g be weakly compatible self maps of a set X. If T and g have a unique point of coincidence $w = Tx = gx$, then w is the unique common fixed point of T and g.*

We add the following two lemmas that will be, as in the paper [8], useful in the proofs of some results. Both these lemmas are used to prove the Cauchyness of the Picard sequence $x_n = Tx_{n-1}, n \in \mathbb{N}$ or the Jungck's sequence $Tx_n = gx_{n+1}, n \in \mathbb{N}$, where $x_0 \in X$ is a given point in a metric space X and $T, g : X \to X$ with $TX \subseteq gX$.

Lemma 1.1. *([8], [10]) Let $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a Picard sequence in a metric space (X, d) such that $d(x_{n+1}, x_n) < d(x_n, x_{n-1})$ is satisfied for all $n \in \mathbb{N}$. Then $x_n \neq x_m$ whenever $n \neq m$.*

Lemma 1.2. *([8], [10]) Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1})$ tends to 0 as $n \to +\infty$. If $\{x_n\}$ is not a Cauchy sequence in X, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following sequences tend to ε from above, as $k \to +\infty$:*

$$\left\{d\left(x_{n(k)}, x_{m(k)}\right)\right\}, \left\{d\left(x_{n(k)}, x_{m(k)-1}\right)\right\}, \left\{d\left(x_{n(k)+1}, x_{m(k)}\right)\right\},$$

$$\left\{d\left(x_{n(k)+1}, x_{m(k)-1}\right)\right\}, \left\{d\left(x_{n(k)+1}, x_{m(k)+1}\right)\right\}, ...$$

2. Improved results

In the paper [3], the authors defined the $F-g-$ contraction, formulated and proved the corresponding theorem. Thus they generalized the result of D. Wardowski from 2012. They are like D. Wardowski used all three introduced properties **F1**, **F2** and **F3**. In this note, we will significantly improve their approach by using only the **F1** property of the function F. For our correction, we used the two mentioned Lemmas and the main property of a strictly increasing function: F has left and right limes at every point a from $(0, +\infty)$ and for them it is fulfilled: $F(a-0) \leq F(a) \leq F(a+0)$.

Definition 2.1. [3] *Let $g : X \to X$ be any mapping and $F \in \mathcal{F}$. A mapping $T : X \to X$ is said to be an $F-g-$contraction if there exists $\tau > 0$ such that*

$$\tau + F(d(Tx, Ty)) \leq F(d(gx, gy)) \tag{1}$$

for all $x, y \in X$ with $gx \neq gy$ and $Tx \neq Ty$.

Theorem 2.1. [3] *Let (X, d) be a metric space, $g : X \to X$ be a mapping and $T : X \to X$ be an $F-g-$contraction such that $T(X) \subseteq g(X)$. If either (X, d) is complete with T and g as continuous and commuting mappings on X or $g(X)$ is complete then g and T have a coincidence point $x \in X$ with the unique point of coincidence gx.*

Important Notice. Let us first point out the following: since it follows from (1) that $d(Tx,Ty) < d(gx,gy)$ whenever Tx is different from Ty (which is equivalent to $d(Tx,Ty) > 0$) this is the condition gx different from gy in Definition 1. redundant. That obviously follows from the Remark 3.7. authors in [3]. Note that this catch gx, unlike gy, is redundant in all their examples before Theorem 3.9. from [3]. Our correction of their Theorem 3.9. is first given by a new formulation.

Theorem 2.2. *Let (X,d) be a metric space, $g : X \to .X$ be a mapping and $T : X \to .X$ be an $F-g-$contraction so that F has only the property **F1** and $T(X) \subseteq g(X)$. Then T and g have a unique point of coincidence. Moreover if T and g are weakly compatible mappings then they have a unique common fixed point which is in fact equal to the resulting unique point of coincidence.*

Proof. First, in the first step, we prove the uniqueness of the existence of a possible point of coincidence. Let there be two distinct points of coincidence of the mappings T and g. Let's denote them by y_1 and y_2. This means that there are two different coincidence points x_1 and x_2 such that $y_1 = Tx_1 = gx_1$ and $y_2 = Tx_2 = gx_2$. Now according to the already mentioned strict inequality $d(Tx,Ty) < d(gx,gy)$ whenever $Tx \neq Ty$, let's obtain that $d(y_1,y_2) < d(y_1,y_2)$, which is a contradiction.

In the second step, we prove the existence of a point of coincidence. Let x_0 be an arbitrary point in X. Now we construct the sequence $Tx_n = gx_{n+1}, n = 0,1,2...$This is possible because TX is a subset of gX. Denote by $y_n = Tx_n = gx_{n+1}$. Let's first prove that $d(y_{n+1},y_n) < d(y_n,y_{n-1})$ for all $n = 1,2,...$ Indeed it follows due to $d(Tx,Ty) < d(gx,gy)$ by putting $x = x_{n+1}, y = x_n$. Further, according to the Lemma 2. we get that y_n is different y_m whenever n is different from m.

If $y_n = y_{n+1}$ for some n then $Tx_{n+1} = gx_{n+1}$ is obviously the unique point of coincidence. So let's assume that y_n is different from y_{n+1} for every n. Since we obtained that the sequence $d(y_{n+1}.y_n)$ is decreasing, it converges to some non-negative number δ. If we assume that $\delta > 0$, then by moving to the limes in the inequality

$$\tau + F\left(d\left(y_{n+1},y_n\right)\right) \leq F\left(d\left(y_n,y_{n-1}\right)\right), \tag{2}$$

when n tends to $+\infty$ and using the property of the function F about the ratio of the left and right limes, we get $\tau + F(\delta) \leq F(\delta)$, which is a contradiction with $\tau > 0$. Now the conditions have been met that Lemma 3 can be applied to prove the Cauchyness of the sequence y_n. Namely, by putting in (1) $x = x_{n(k)}, y = x_{m(k)}$ and switching to limes when $k \to +\infty$, we get $\tau + F(\varepsilon^+) \leq F(\varepsilon^+)$, which is a contradiction with $\tau > 0$. Again we applied the property of the function F about the right limes. So we got that the sequence $y_n = Tx_n = gx_{n+1}$ is a Cauchy sequence.

Since the metric space (X,d) is complete (the first possibility in the assumption), then the constructed Cauchy sequence $y_n = Tx_n = gx_{n+1}$ converges to some point x^* from X. Using the fact that the mappings T and g are continuous, then just like in [3] we prove that $Tx^* = gx^*$, that is, that $Tx^* = gx^*$ is the point of coincidence for the mapping of T and g (and this is the only one that was proved at the beginning as a first step). If we assume another possibility, i.e, that $g(X)$ is complete, then there exists x^* from X such that $T\,x_n = gx_{n+1}$ converges to gx^*. Further using the already mentioned condition $d(Tx,Ty) < d(gx,gy)$ whenever Tx is different from Ty we get that $d(Tx_n,Tx^*) < d(gx_n,gx^*)$. From which it follows that Tx_n tends to Tx^* when n tends to $+\infty$. Since we are dealing with equal sequences Tx_n and gx_{n+1}, we get that $Tx^* = gx^*$is a point of coincidence for the mappings T and g (the only one). If, in each of the two separate cases, weak compatibility of mapping T and g is assumed, then according to Proposition 1. we get that T and g have a unique common fixed point $Tx^* = gx^*$. The proof of Theorem is complete. □

An important note. Note that in the paper [4] on the so-called F_w-contractions for a mapping $F : (0, +\infty) \to \mathbb{R}$, all three properties **F1)**, **F2)** and **F3)** are also assumed. Using our approach presented in the papers [8] and [10], only the property **F1)** can be assumed for the function F. This then significantly improves the results presented in [3] and [4].

3. About $F-g$ weak contractions

Let (X, d) be the complete metric space provided with the metric d. And let T and g be mappings of the set X into itself. Let it be further assumed that TX is a subset of gX. Using the mappings T and g, consider the following two numerical sets:

$$M^1_{T,g}(x,y) = \max\left\{d\left(gx,gy\right), d\left(Tx,gx\right), d\left(Ty,gy\right), \frac{d\left(Tx,gy\right)+d\left(Ty,gx\right)}{2}\right\} \tag{3}$$

and

$$M^2_{T,g}(x,y) = \max\left\{d\left(gx,gy\right), \frac{d\left(Tx,gx\right)+d\left(Ty,gy\right)}{2}, \frac{d\left(Tx,gy\right)+d\left(Ty,gx\right)}{2}\right\} \tag{4}$$

Obviously $M^2_{T,g}(x,y) \le M^1_{T,g}(x,y)$. Further, let F be a strictly increasing mapping of the set $(0, +\infty)$ into $\mathbb{R}$. We have the following definition:

In this part of the paper, the function F that maps the set of positive real numbers $(0, +\infty)$ to the set of real numbers $\mathbb{R}$ will be strictly increasing, i.e. it will satisfy property **F1** from Wardowski's conditions. We will assume this throughout this paper.

Definition 3.2. *Let (X, d) be a metric space and let T, g be mappings from X in X with $TX \subseteq gX$. A map $T : X \to X$ is said to be an $F-g$ weak contraction on (X, d) if there exist strongly increasing function F from $(0, +\infty)$ in $\mathbb{R}$, $\tau > 0$ such that, for all $x, y \in X$ satisfying $d\left(Tx, Ty\right) > 0$, the following holds:*

$$\tau + F\left(d\left(Tx,Ty\right)\right) \le F\left(M^1_{T,g}(x,y)\right). \tag{5}$$

We now formulate the corresponding result.

Theorem 3.3. *Let (X, d) be a metric space, $g : X \to X$ be a mapping and $T : X \to X$ be an $F-g-$contraction so that F has only the property **F1** and $T(X)\subseteq g(X)$. Then T and g have a unique point of coincidence. Moreover, if T and g are weakly compatible mappings then they have a unique common fixed point which is in fact equal to the resulting unique point of coincidence.*

Proof. In a similar way as in the proof of Theorem 5. in this case we first prove the uniqueness of the point of coincidence if it exists. If there are two distinct points of coincidence y_1 and y_2 then there are two distinct coincidence points x_1 and x_2 such that $y_1 = Tx_1 = gx_1$ and $y_2 = Tx_2 = gx_2$. First let's calculate:

$$M^1_{T,g}(x_1,x_2)$$

$$= \max\left\{d\left(gx_1, gx_2\right), d\left(Tx_1, gx_1\right), d\left(Tx_2, gx_2\right), \frac{d\left(Tx_1, gx_2\right) + d\left(Tx_2, gx_1\right)}{2}\right\}$$

$$= \max\left\{d\left(y_1, y_2\right), d\left(y_1, y_1\right), d\left(y_2, y_2\right), \frac{d\left(y_1, y_2\right) + d\left(y_2, y_1\right)}{2}\right\} = d\left(y_1, y_2\right). \quad (6)$$

Now, putting in (5) $x = x_1, y = x_2$ we get

$$\tau + F\left(d\left(Tx_1, Tx_2\right)\right) \leq F\left(M^1_{T,g}(x_1, x_2)\right), \quad (7)$$

or equivalently,

$$\tau + F\left(d\left(y_1, y_2\right)\right) \leq F\left(d\left(y_1, y_2\right)\right). \quad (8)$$

We get the contradiction with $\tau > 0$. Therefore, the point of coincidence of T and g is a unique if it is exists.

In a similar way as in the proof of the corrected result in the previous section, we prove that the sequence $y_n = Tx_n = gx_{n+1}$ is a Cauchy sequence. Let firstly the metric space (X, d) is a complete. Then the sequence y_n converges to some point $u \in X$. And if we assume continuity and commutativity of mappings T and g we have:

$$Tu = T\left(\lim_{n\to+\infty} gx_n\right) = \lim_{n\to+\infty} Tgx_n = \lim_{n\to+\infty} gTx_n = g\left(\lim_{n\to+\infty} Tx_n\right) = gu. \quad (9)$$

Hence, $Tu = gu$ is a unique point of coincidence of the pair (T, g).

If now, we suppose that (gX, d) is a complete and that the mapping F is continuous we have first that the sequence $y_n = Tx_n = gx_n$ converges to gu for some $u \in X$. Since, we can suppose that $y_n \neq y_m$ whenever $n \neq m$. In this case we also can suppose that $u \neq y_n$ for each $n \in \mathbb{N}$. Further, the condition (5) implies that

$$d\left(Tx, Ty\right) < M^1_{T,g}(x, y), \quad (10)$$

whenever $d(Tx, Ty) > 0$. Putting in (5) $x = x_n, y = u$ we get

$$\tau + F(d\left(Tx_n, Tu\right)) \leq F(M^1_{T,g}(x_n, u))$$

$$= F(\max\left\{d\left(gx_n, gu\right), d\left(Tx_n, gx_n\right), d\left(Tu, gu\right), \frac{d\left(Tx_n, gu\right) + d\left(Tu, gx_n\right)}{2}\right\}). \quad (11)$$

The last relation and the continuous of the function F imply that

$$\tau + F(d(gu, Tu)) \leq F(d(gu, Tu)), \quad (12)$$

what contradicts to $\tau > 0$, if $gu \neq Tu$. □

3.1. **Some conclusions.** The mappings T and g have a unique point of coincidence if at least one of the following two cases is fulfilled:

1. The metric space (X, d) is complete and the mappings T and g are continuous and commutative;

2. The metric space (gX, d) is complete and the mapping F is continuous.

If in each of the cases the **1**. and **2.** the mappings T and g are weak compatible, then they have an obtained point of coincidene for a unique common fixed point.

Of course, in each case of mapping T and g have a unique point of coincidence. Also in each of the cases **1.** and **2**. if T and g are weakly compatible mappings, then a unique point of coincidence is simultaneously obtained and their common fixed point is unique.

Remark 1. *Theorem 6. significantly generalizes the results from* [12] *according to the following two grounds:*

a) Instead of the identical mapping i_X, it was switched to the more efficient mapping g from X to X;

b) Instead of all three properties **F1)**, **F2)** and **F3)** of mapping F from $(0, +\infty)$ to $\mathbb{R}$ we assumed-used only property **F1**). This provided us with the main property of an increasing (non-decreasing) function about the existence of left and right limits at each point a from $(0, +\infty)$.

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