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UNIQUENESS OF MEROMORPHIC FUNCTIONS THAT SHARE VALUES IM WITH THEIR DERIVATIVES

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Abstract. In this paper, we investigate the problem of uniqueness of meromorphic functions sharing two values IM(Ignoring Multiplicities) with their derivatives. Some examples are provided to show the sharpness of the result. The obtained result improves and generalizes the corresponding result from [S. Chen and A. Xu, Uniqueness of derivatives and shifts of meromorphic functions, Comput. Methods Funct. Theory, **22** (2022), 197-205].

1. Introduction, Definitions and Results

Let $f(z)$ be a nonconstant meromorphic function in the whole complex plane. We shall use the following standard notations of Nevanlinna theory, for instance $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$ (see [5, 10, 15]). We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $(r \to \infty, r \notin E)$, where E denotes any set of positive real numbers having finite linear measure. The hyper order of f is defined as follows

$$\rho_2(f) = limsup_{r\to\infty}\frac{log^+log^+T(r,f)}{logr}.$$

Let k be a positive integer or infinity and $a \in \mathbb{C}\cup\{\infty\}$. Set $E(a,f) = \{z : f(z)-a = 0\}$, where a zero with multiplicity k is counted k times. If the zeros are counted only once, then we denote the set by $\overline{E}(a,f)$. Let f and g be two non-constant meromorphic functions. If $E(a,f) = E(a,g)$, then we say that f and g share the value a CM (Counting Multiplicities). If $\overline{E}(a,f) = \overline{E}(a,g)$, then we say that f and g share the value a IM (Ignoring Multiplicities). We denote by $E_{k)}(a,f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a,f)$ the set of distinct a-points of f with multiplicities not exceeding k. In addition, we need the following definitions.

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Definition 1.1. [8] *Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f)$ the counting function of simple a-points of f and $\overline{N}(r, a; f)$ denote the corresponding reduced counting function. For a positive integer k we denote by $N_{k)}(r, a; f)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than k. By $\overline{N}_{k)}(r, a; f)$ we denote the corresponding reduced counting function. Let $N_{(k}(r; a; f)$ be the counting function of zeros of $f - a$ with multiplicity at least k and $\overline{N}_{(k}(r, a; f)$ the corresponding one for which multiplicity is not counted.*

Definition 1.2. [9] *Let k be a positive integer or infinity. We denote by $N_k(r, a; f)$ the counting function of a-points of f, where an a-points of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then*

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}_{(2}(r, a; f) + ... + \overline{N}_{(k}(r, a; f).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

As we know, Nevanlinna theory plays an important role in the study of complex differential equations and complex difference equations, (see[1, 2, 10, 14]). Recently, many papers have focused on the study of complex differential-difference equations (called DDE for short), (see [6, 11]) and so on.
For the DDE $f'(z) = f(z + c)$, X. Qi, L. Yang [12] considered the uniqueness problem between $f'(z)$ and $f(z + c)$ of meromorphic functions $f(z)$ in view of Nevanlinna theory and obtained the following result.

Theorem A. [12] *Let $f(z)$ be a transcendental entire function of finite order and $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z + c)$ share $0, a$ CM, then $f'(z) = f(z + c)$.*

To further improve Theorem A, a more general question could be posed as follows: could we determine the relationship between the k-th derivative $f^{(k)}(z)$ and the shift $f(z + c)$ of a meromorphic (or entire) function $f(z)$ under more general sharing value conditions?
In 2021, S. Chen and A. Xu [3] proved a uniqueness theorem about the k-th derivative $f^{(k)}(z)$ and the shift $f(z+c)$ of a meromorphic function $f(z)$ with two CM sharing values and one IM sharing value, which greatly generalizes and improves Theorem A. Their result can be stated as follows.

Theorem B. [3] *Let $f(z)$ be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$, c be a non-zero finite complex number, and k be a positive integer. If $f^{(k)}(z)$ and $f(z+c)$ share $0, \infty$ CM and 1 IM, then $f^{(k)}(z) = f(z + c)$.*

Regarding Theorem B it is natural to ask the following questions.

Question 1.1. *What will happen if we replace the function $f^{(k)}$ by the function $L(f)$, where $L(f)$ is defined as follows $L(f) = a_k f^{(k)} + a_{k-1} f^{(k-1)} + ... + a_1 f' + a_0 f, a_k(\neq 0), a_{k-1}, ..., a_1, a_0 \in \mathbb{C}$.*

Question 1.2. *Is it possible to relax the nature of sharing in Theorem B?*

In this paper, we continue to study the above questions. We shall prove a uniqueness theorem with two IM sharing, which would generalize and improve Theorem B. The following theorem is the main result of the paper.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function of hyper order strictly less than 1 and let a, b be two distinct finite values. If $f(z+c)$ and $L(f)$ share a, b IM and $\overline{N}(r, a; f) + \overline{N}(r, \infty; f) = S(r, f)$, then $f(z + c) = L(f)$.*

Remark 1. *It is easy to see that the condition $\overline{N}(r, a; f) + \overline{N}(r, \infty; f) = S(r, f)$, in Theorem 1.1 is necessary by the following examples.*

Example 1.1. *Let* $f(z) = \frac{2}{1-e^{-2z}}$ *and* $a = 0, b = 1$, *then for* $c = \pi i$, $f(z+c)$ *and* $L(f)$ *share the value* $0, 1$ *IM but* $f(z+c) \not\equiv L(f)$ *because* $\overline{N}(r, a; f) + \overline{N}(r, \infty; f) \neq S(r, f)$.

Example 1.2. *Let* $f(z) = sinz$, *then for suitable values of* a, b, c, $f(z+c)$ *and* $L(f)$ *share the value* a, b *IM but* $f(z+c) \not\equiv L(f)$.

Remark 2. *From the condition of Theorem 1.1 and* $f(z+c) \not\equiv L(f)$, *we have* $S(r, f(z+c)) = S(r, L(f))$. *In fact, since* $\overline{N}(r, \infty; f(z+c)) = S(r, f(z+c))$, *this means that*

$$N(r, \infty; f(z+c)) \leq m\overline{N}(r, \infty; f(z+c)) + S(r, f(z+c)) = S(r, f(z+c)), \tag{1.1}$$

where m *is the maximum order of all poles of* $f(z+c)$. *If* $f(z+c) \not\equiv L(f)$, *then by Lemma 2.3 below and (1.1), it follows that*

$$\begin{aligned}
T(r, f(z+c)) &\leq \overline{N}(r, a; f(z+c)) + \overline{N}(r, b; f(z+c)) + \overline{N}(r, \infty; f(z+c)) + S(r, f(z+c)) \\
&\leq \overline{N}(r, a; L(f)) + \overline{N}(r, b; L(f) + S(r, f(z+c)) \\
&\leq \overline{N}(r, 0; (f(z+c) - L(f))) + S(r, f(z+c)) \\
&\leq T(r, (f(z+c) - L(f))) + S(r, f(z+c)) \\
&\leq m(r, (f(z+c) - L(f))) + S(r, f(z+c)) \\
&\leq m\left(r, \frac{(f(z+c) - L(f))}{f(z+c)}\right) + m(r, f(z+c)) + S(r, f(z+c)) \\
&\leq T(r, f(z+c)) + S(r, f(z+c)),
\end{aligned}$$

which implies that

$$2T(r, L(f)) \quad \geq \quad \overline{N}(r, a; L(f)) + \overline{N}(r, b; L(f)) = T(r, f(z+c)) + S(r, f(z+c)). \tag{1.2}$$

On the other hand, with Lemma 2.3 below and the definition of $L(f)$, *we have*

$$T(r, L(f)) \leq (k+1)T(r, f(z+c)) + S(r, f(z+c)). \tag{1.3}$$

By combining (1.2) and (1.3), we get $S(r, f(z+c)) = S(r, L(f))$. *Again we know that* $S(r, f(z+c)) = S(r, f(z))$. *Therefore we use* $S(r, f(z+c)) = S(r, L(f)) = S(r, f(z))$. *For simplicity we take* $f(z+c) = f_c(z)$.

Definition 1.3. [15] *Let* $f(z)$ *be a meromorphic function in the complex plane and* a *be any finite value. If* $f(z) - a$ *has no zeros, then* a *is called a Picard value of* $f(z)$.

2. Lemmas

Lemma 2.1. [7] *Let* $f(z)$ *be a meromorphic function of hyper order strictly less than 1. Then*
$m(r, \frac{f(z)}{f(z+c)}) + m(r, \frac{f(z+c)}{f(z)}) = S(r, f)$ *and* $m(r, \frac{L(f)}{f-a}) = S(r, f)$, *where* a *is a constant.*

Lemma 2.2. [7] *Let* $T : [0, +\infty) \to [0, +\infty)$ *be a non-dreasing continuous function, and let* $s \in (0, +\infty)$. *If the hyper order of* T *is strickly less than 1, i.e.,*

$$limsup_{r\to\infty} \frac{loglogT(r)}{logr} = \epsilon < 1,$$

and $\delta \in (0, 1-\epsilon)$, *then*

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{\delta}}\right),$$

where r *runs to infinity outside of a set of finite logarithmic measure.*

Lemma 2.3. [4]*Let $f(z)$ be a meromorphic function of hyper order strictly less than 1, then we have*

$$N(r, f(z+c)) = N(r, f) + S(r, f)$$

and

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.4. [15]*Let $f(z)$ be a non-constant meromorphic function, $a \in \mathbb{C}$, and k be a positive integer. Then*

$$\overline{N}(r, a; f) \leq \frac{k}{k+1}\overline{N}_{k)}(r, a; f) + \frac{1}{k+1}T(r, f) + O(1).$$

Lemma 2.5. [15] *Suppose that $f(z)$ is a non-constant meromorphic function and $P(f) = a_p f^p + a_{p-1}f^{p-1} + ... + a_0 (a_p \neq 0)$ is a polynomial in $f(z)$ with degree p and coefficients $a_i (i = 0, 1, ..., p)$ are constants, suppose furthermore that $b_j (j = 0, 1, ..., q)$ are distinct finite values. Then*

$$m\left(r, \frac{P(f)f'}{(f-b_1)(f-b_2)...(f-b_q)}\right) = S(r, f).$$

Lemma 2.6. [15]*Let $f(z)$ be a non-constant meromorphic function and a, b be two distict finite values. If a and b are Picard values of $f(z)$, then $f(z) = \frac{ae^{h(z)}-b}{e^{h(z)}-1}$, where $h(z)$ is a non-constant entire function.*

Lemma 2.7. *Let $f(z)$ be a meromorphic function of hyper order strictly less than 1, and let a, b be two distict finite values. Suppose that f_c and $L(f)(\not\equiv 0)$ share a, b IM, and $\overline{N}(r, a; f) + \overline{N}(r, \infty; f) = S(r, f)$. And suppose furthermore that $f_c \neq L(f)$. Then the following holds.*
(i) $T(r, f_c) = \overline{N}(r, b; f_c) + S(r, f)$, $T(r, L(f)) = \overline{N}(r, b; L(f)) + S(r, f)$. Moreover, we have $T(r, L(f)) = T(r, f_c) + S(r, f)$.
(ii) $T(r, f_c) = \overline{N}(r, d; f_c) + S(r, f)$, $T(r, L(f)) = \overline{N}(r, d; L(f)) + S(r, f)$, where $d(\neq a, b) \in \mathbb{C}$.
(iii) $N^*(r, a) + N^*(r, b) = S(r, f)$, where $N^*(r, a)$ is the counting function of the multiple common zeros of $f_c - a$ and $L(f) - a$, which counts multiplicities according to the minor one, notation $N^*(r, b)$ can be similarly defined.
(iv) $\overline{N}_{(2}(r, b; f_c) = S(r, f)$, $\overline{N}_{(2}(r, b; L(f)) = S(r, f)$.
(v) $\overline{N}(r, b; f_c) - N_E(r, b) = S(r, f)$, where $N_E(r, b)$ is the reduced counting function of the common zeros of $f_c - a$ and $L(f) - a$ with the same multiplicities.
(vi) $\overline{N}(r, 0; f_c') = S(r, f)$, $\overline{N}(r, 0; L'(f)) = S(r, f)$.

Proof. (i)
The assumption that $\overline{N}(r, a; f_c) + \overline{N}(r, \infty; f_c) = S(r, f)$, together with the second main theorem, means

$$\begin{aligned} T(r, f_c) &\leq \overline{N}(r, a; f_c) + \overline{N}(r, b; f_c) + \overline{N}(r, \infty; f_c) + S(r, f) \\ &\leq \overline{N}(r, b; f_c) + S(r, f) \\ &\leq T(r, f_c) + S(r, f), \end{aligned}$$

which implies that

$$T(r, f_c) = \overline{N}(r, b; f_c) + S(r, f). \tag{2.1}$$

From the assumption, we have

$$\overline{N}(r, a; L(f)) + \overline{N}(r, \infty; L(f)) = S(r, f),$$

and

$$\overline{N}(r, b; f_c) = \overline{N}(r, b; L(f)). \tag{2.2}$$

Similarly, we have

$$T(r,L(f))=\overline{N}(r,b;L(f))+S(r,f). \tag{2.3}$$

From (2.1) to (2.3), it follows that

$$T(r,L(f))=T(r,f_c)+S(r,f).$$

(ii)
By the second main theorem, the assumption that $\overline{N}(r,a;f_c)+\overline{N}(r,\infty;f_c)=S(r,f)$, and the conclusion (i), we have

$$\begin{aligned}2T(r,f_c) &\leq \overline{N}(r,a;f_c)+\overline{N}(r,b;f_c)+\overline{N}(r,d;f_c)+\overline{N}(r,\infty;f_c)+S(r,f)\\ &\leq T(r,f_c)+\overline{N}(r,d;f_c)+S(r,f)\\ &\leq 2T(r,f_c)+S(r,f),\end{aligned}$$

which implies that

$$T(r,f_c)=\overline{N}(r,d;f_c)+S(r,f).$$

Similarly, we have

$$T(r,L(f))=\overline{N}(r,d;L(f))+S(r,f).$$

(iii)
Let us denote

$$g(z)=\frac{f_c'(f_c-L(f))}{(f_c-a)(f_c-b)}. \tag{2.4}$$

Then, by (1.1) and the value sharing assumption, we know $N(r,g)=S(r,f)$. Next, we write (2.4) in the form

$$g(z)=\frac{f_c'f_c}{(f_c-a)(f_c-b)}\frac{f_c-L(f)}{f_c}.$$

Using Lemma 2.5, we get

$$m\left(r,\frac{f_c'f_c}{(f_c-a)(f_c-b)}\right)=S(r,f).$$

By Lemma 2.1 we easily obtain that

$$m\left(r,\frac{f_c-L(f)}{f_c}\right)=S(r,f).$$

Thus,

$$T(r,g)=m(r,g)=S(r,f). \tag{2.5}$$

On the other hand, let z_0 be a multiple common zeros of f_c-a (or f_c-b) and $L(f)-a$ (or $L(f)-b$) with multiplicities p and q $(p\geq 2, q\geq 2)$ respectively. By a simple computation, we know z_0 is a zero of $g(z)$ with multiplicity $min\{p,q\}-1$ (≥ 1) at least. Therefore, we have

$$N^*(r,a)\leq 2N(r,0;g)+S(r,f)\leq 2T(r,g)+S(r,f)=S(r,f),$$

$$N^*(r,b)=S(r,f),$$

and so

$$N^*(r,a)+N^*(r,b)=S(r,f).$$

(iv)
Combining Lemma 2.4 and the conclusion (i), we have

$$T(r,f_c)+S(r,f)=\overline{N}(r,b;f_c)\leq\frac{1}{2}\overline{N}_{1)}(r,b;f_c)+\frac{1}{2}T(r,f_c)+S(r,f),$$

which implies

$$T(r,f_c) \leq \overline{N}_{1)}(r,b;f_c) + S(r,f) \leq T(r,f_c) + S(r,f).$$

This together with the conclusion (i), implies that

$$T(r,f_c) = \overline{N}_{1)}(r,b;f_c) + S(r,f) = \overline{N}(r,b;f_c) + S(r,f).$$

and so

$$\overline{N}_{(2}(r,b;f_c) = S(r,f),$$

and similarly, we have

$$\overline{N}_{(2}(r,b;L(f)) = S(r,f).$$

(v)
We denote by $N_E^{1)}(r,b)$ the reduced counting function of the common simple zeros of $f_c - b$ and $L(f) - b$. Using the conclusion (iv) and the value sharing assumption, we obtain

$$\overline{N}(r,b;f_c) \geq N_E^{1)}(r,b) \geq \overline{N}_{1)}(r,b;f_c) - \overline{N}_{(2}(r,b;L(f)) = \overline{N}(r,b;f_c) + S(r,f),$$

which means

$$\overline{N}(r,b;f_c) = N_E^{1)}(r,b) + S(r,f). \tag{2.6}$$

On the other hand, by the conclusion (iii), we have

$$N_E(r,b) - N_E^{1)}(r,b) \leq N^*(r,b) = S(r,f)$$

and so

$$N_E(r,b) = N_E^{1)}(r,b) + S(r,f). \tag{2.7}$$

From (2.6) and (2.7), we see

$$\overline{N}(r,b;f_c) - N_E(r,b) = S(r,f).$$

(vi)
We denote by $N_0(r,0;f_c')$ the counting function of zeros of f_c' but not the zeros of $f_c - a$ and $f_c - b$; $N_0(r,0;L'(f))$ is defined similarly. Then by the second main theorem and the conclusion (i), it follows that

$$\begin{aligned} T(r,f_c) &\leq \overline{N}(r,a;f_c) + \overline{N}(r,b;f_c) + \overline{N}(r,\infty;f_c) - N_0(r,0;f_c') + S(r,f) \\ &\leq T(r,f_c) - N_0(r,0;f_c') + S(r,f). \end{aligned}$$

Thus,

$$N_0(r,0;f_c') = S(r,f). \tag{2.8}$$

and similarly,

$$N_0(r,0;L'(f)) = S(r,f). \tag{2.9}$$

Now by (2.8), (2.9), the conclusion (iv) and the assumption that $\overline{N}(r,a;f_c) = \overline{N}(r,a;L(f)) = S(r,f)$, we have

$$\overline{N}(r,0;f_c') = S(r,f), \overline{N}(r,0;L'(f)) = S(r,f).$$

This completes the proof. □

Lemma 2.8. [15] *Suppose that $f_j(z)$ $(j = 1,2,...,n)(n \geq 2)$ are meromorphic functions and $g_j(z)$ $(j = 1,2,...,n)$ are entire functions satisfying the following conditions.*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} = 0.$

(2) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$.

(3) For $1 \leq j \leq n$, $1 \leq h < k \leq n$

$$T(r,f_j) = o\{T(r,e^{g_h - g_k})\}, r \to \infty, r \notin E,$$

where $E \subset (1, \infty)$ is of finite linear measure.
Then $f_j(z) = 0$.

Lemma 2.9. *Let f_c be a transcendental entire function of hyper order strictly less than 1, and let a, b be two distinct finite values. If f_c and $L(f)(\not\equiv 0)$ share b IM, and a is a Picard value of f_c and $L(f)$. Then $f_c = L(f)$.*

Proof. Since a is a Picard value of f_c and $L(f)$, we have

$$f_c = e^{h(z+c)} + a,$$

where $h(z+c)$ is a non-constant entire function of order less than 1. Moreover, we get

$$\frac{L(f) - a}{f_c - a} = \frac{\sum_{i=0}^{k} B_i e^{h(z+c)} - a}{e^{h(z+c)}},$$

where $B_i = B_i(h^{(1)}, h^{(2)}, ..., h^{(k)})$ is a Bell polynomial.

$$= \sum_{i=0}^{k} B_i - a e^{-h(z+c)} = e^{Q(z+c)}, \tag{2.10}$$

where $Q(z+c)$ is entire.
If $Q(z+c)$ is a constant, then

$$\frac{L(f) - a}{f_c - a} = A,$$

where A is a non-zero constant. Since f_c and $L(f)$ share b IM, which shows $A = 1$.
Hence $f_c = L(f)$.
If $Q(z+c)$ is not a constant, then it follows from (2.10) that

$$\sum_{i=0}^{k} B_i - a e^{-h(z+c)} - e^{Q(z+c)} = 0. \tag{2.11}$$

If $Q(z+c) + h(z+c) = C$, where C is a constant. Then from (2.11) can be rewritten as

$$\sum_{i=0}^{k} B_i = (e^C + a) e^{-h(z+c)},$$

which is impossible, since $\sum_{i=0}^{k} B_i \neq 0$ and $h(z+c)$ is not a constant. Thus $Q(z+c)+h(z+c)$ is not a constant, and so $h(z+c)$, $Q(z+c)$ and $h(z+c) + Q(z+c)$ are not constants. Applying Lemma 2.8 to (2.11), we get a contradiction.
This completes the proof. □

3. Proof of Theorem 1.1

Proof. Assume to the contrary that $f_c \not\equiv L(f)$. Set

$$F(z) = \frac{f_c - a}{f_c - b}, \quad G(z) = \frac{L(f) - a}{L(f) - b}. \tag{3.1}$$

Then, we have

$$T(r, F) = T(r, f_c) + S(r, f), \quad T(r, G) = T(r, L(f)) + S(r, f),$$

and by the conclusion (i) of Lemma 2.7,

$$S(r, F) = S(r, f), \quad S(r, G) = S(r, f).$$

Since f_c and $L(f)$ share a, b IM, $F(z)$ and $G(z)$ share $0, \infty$ IM. Further, the conclusions (i) and (iv) of Lemma 2.7 imply that

$$T(r,F)=\overline{N}(r,\infty;F)+S(r,f),\quad T(r,G)=\overline{N}(r,\infty;G)+S(r,f) \tag{3.2}$$

and

$$\overline{N}_{(2}(r,\infty;F)+\overline{N}_{(2}(r,\infty;G)=S(r,f). \tag{3.3}$$

Moreover from (3.1), we have

$$F'=\frac{(a-b)f_c'}{(f_c-b)^2},\quad G'=\frac{(a-b)L'(f)}{(L(f)-b)^2}.$$

This together with the conclusion (vi) of Lemma 2.7 lead to

$$\overline{N}(r,0;F')\le\overline{N}(r,0;f_c')+S(r,f)=S(r,f), \tag{3.4}$$

$$\overline{N}(r,0;G')\le\overline{N}(r,0;L'(f))+S(r,f)=S(r,f).$$

Set

$$H(z)=\frac{F''}{F'}-\frac{G''}{G'}. \tag{3.5}$$

Let z_0 be a pole of $F(z)$ and $G(z)$ with the same multiplicities k(by (3.1), we know z_0 is a zero of f_c-b and $L(f)-b$ with the same multiplicities $k(\geq 1)$). Then, we obtain

$$F(z)=\frac{a_{-k}}{(z-z_0)^k}+\frac{a_{-(k-1)}}{(z-z_0)^{k-1}}+...+a_0+a_1(z-z_0)+a_2(z-z_0)^2+...$$

and

$$G(z)=\frac{b_{-k}}{(z-z_0)^k}+\frac{b_{-(k-1)}}{(z-z_0)^{k-1}}+...+b_0+b_1(z-z_0)+b_2(z-z_0)^2+...,$$

where $a_{-k}b_{-k}\neq 0$. A simple computation shows that

$$H(z)=\frac{k-1}{k}\left(\frac{a_{-(k-1)}}{a_{-k}}-\frac{b_{-(k-1)}}{b_{-k}}\right)+A_1(z-z_0)+A_2(z-z_0)^2+..., \tag{3.6}$$

which means that z_0 is not a pole of $H(z)$. Using (3.1), (3.4) and the conclusion (v) of Lemma 2.7, we have

$$\begin{aligned} N(r,\infty;H) &\le \overline{N}(r,\infty;F)-N_E(r,\infty)+\overline{N}(r,\infty;G)-N_E(r,\infty)\\ &\quad +\overline{N}(r,0;F')+\overline{N}(r,0;G')+S(r,f)\\ &= \overline{N}(r,b;f_c)+\overline{N}(r,b;L(f))-2N_E(r,\infty)+S(r,f)=S(r,f), \end{aligned} \tag{3.7}$$

where $N_E(r,\infty)$ is the reduced counting function of the poles of $F(z)$ and $G(z)$ with the same multiplicities. Obiviously applying the Lemma on logarithmic derivative to (3.5), it follows that

$$m(r,H)=S(r,f). \tag{3.8}$$

Thus by (3.7) and (3.8), we obtain

$$T(r,H)=S(r,f). \tag{3.9}$$

In the following, we will discuss two cases.

Case 1. $H(z)\not\equiv 0$. In this case (3.2) and (3.3) imply that $F(z)$ and $G(z)$ have infinitely many simple poles. Let z_0 be a simple pole of $F(z)$ and $G(z)$ then from (3.6), we know z_0 is a zero of $H(z)$. Therefore from (3.3) and (3.9) we have

$$\overline{N}(r,\infty;F)+S(r,f)=\overline{N}(r,\infty;F)-\overline{N}_{(2}(r,\infty;F)\le N(r,0;H)+S(r,f)=S(r,f),$$

which contradicts (3.2).
Case 2. $H(z) \equiv 0$. In this case (3.5) means that

$$F(z) = AG(z) + B, \tag{3.10}$$

where $A(\neq 0)$ and B are two constants.
We claim that a is not a Picard value of f_c and $L(f)$. Otherwise, if a is a Picard value of f_c and $L(f)$, then we see that 0 is a Picard value of $F(z)$ and $G(z)$. Moreover, since $a \neq b$, by (3.1), it shows 1 is a Picard value of $F(z)$ and $G(z)$. Hence 0 and 1 are two Picard values of $F(z)$. From Lemma 2.6 and (3.1), it follows that

$$\frac{f_c - a}{f_c - b} = F(z) = \frac{-1}{e^{h(z)} - 1},$$

which leads to $f_c(z) = (b-a)e^{-h(z)} + a$, where $h(z)$ is a non-constant entire functions and so $f_c(z)$ is entire. By Lemma 2.9, we know $f_c(z) = L(f)$, which contradicts the assumption $f_c(z) \not\equiv L(f)$.
Hence, a is not a Picard value of f_c and $L(f)$, and so 0 is not a Picard value of $F(z)$ and $G(z)$. Further, (3.10) yields that $B = 0$. Thus,

$$F(z) = AG(z). \tag{3.11}$$

Since that $a \neq b$, by (3.1), we have 1 is Picard value of $F(z)$ and $G(z)$. Moreover, invoking $A \neq 0$, from (3.11), we see A is a Picard value of $F(z)$. Therefore,

$$\frac{f_c - a}{f_c - b} \neq A,$$

which implies that $f_c \neq \frac{Ab-a}{A-1}$ (note that $f_c(z) \not\equiv L(f)$, we obtain that $A \neq 1$). It is obvious that

$$\frac{Ab - a}{A - 1} \neq a, \quad \frac{Ab - a}{A - 1} \neq b.$$

Hence, we conclude that $\frac{Ab-a}{A-1}$ is a Picard value of f_c, which contradicts the conclusion (ii) of Lemma 2.7. This completes the proof of the theorem.

□

4. **Conclusion**

In this research article, we have proved a theorem by using the concept of sharing two values IM(Ignoring Multiplicities) and replacing a differential function by its polynomial which improves and generalizes the previous result obtained in [3].

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