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EXPLICIT SOLUTION FOR BACKWARD STOCHASTIC VOLTERRA INTEGRAL EQUATIONS WITH LINEAR TIME DELAYED GENERATORS

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Abstract. This note aims to give an explicit solution for backward stochastic Volterra integral equations with linear time delayed generators. The process Y is expressed by an integral whose kernel is explicitly given. The processes Z is expressed by Hida-Malliavin derivatives involving Y. This paper generalized the work Hu and Oksendal who treat the no delay case.

1. Introduction

Backward stochastic Volterra integral equations (BSVIEs, for short) have been initiated in [10] under Lipschitz condition. This assumption has been relaxed to local Lipschitz condition in [2]. A few years later, the completed theory of backward stochastic Volterra integral equations (BSVIEs, for short) has been introduced by Yong in [16, 15, 14] and references therein. This king of BSDEs has been connected to optimal control problems for controlled Volterra type systems. Recently, in [3], Coulibaly and Aman study under Lipschitz assumption the following BSVIEs with general time delayed generator: For a fixed $T > 0$ and $0 \leq t \leq T$,

$$Y(t) \quad = \quad \xi + \int_t^T f(t, s, Y_s, Z_{t,s})ds - \int_t^T Z(t, s)dW(s), \tag{1.1}$$

where $(Y_s, Z_{t,s}) = (Y(s + u), Z(t + u, s + u))_{-T\leq u\leq 0}$ denoted the past of process (Y, Z) until (t, s). In the special case we can consider the function f of this form: $f(t, s, y_s, z_{t,s}) = \int_{-T}^0 g(t+u, s+u, z(t+u, s+u))\alpha(du)$, where α is called delay probability measure. This type of BSDEs can be viewed as the combinaison of Voltera BSDEs and delayed BSDE introduced in [6, 7]. This study is very interesting since BSVIEs with delayed generator

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are used to model a recursive utility with delay which appears when we face the problem of dynamic modeling of non monotone preferences. For more detail we refer the reader to [5] and reference therein. However, as the counter examples in [3] attest, the existence and uniqueness of the solution is not valid in general. We distinguish two conditions of existence and uniqueness. First, when the delay probability measure is general, then it is necessary and sufficient that, in a certain sense, the Lipschitz constant of the generator or the terminal time be small enough. But in the case of special delay measure (it is supported by $[-\gamma, 0]$, for $\gamma > 0$ sufficiently small), the existence and uniqueness result hold for all Lipschitz constant and terminal time.

In this paper our study concern a linear BSVIE with time delayed generator which is a special case of BSVIE (1.1). Let $\{F(t), 0 \leq t \leq T\}$ be a given stochastic process, not necessarily adapted, $(G(t,s), -T \leq t \leq s \leq T)$ and $(g(t,s), -T \leq t \leq s \leq T)$ be given processes with values in $\mathbb{R}$ and α a delay probability measure defined in $([-T,0], \mathcal{B}([-T,0]))$. We consider the following I-type delayed BSVIE:

$$\begin{aligned} Y(t) &= F(t) + \int_t^T \int_{-T}^0 [G(t+u, s+u)Y(s+u) + g(t+u, s+u)Z(t+u, s+u)]\alpha(du)ds \\ &\quad - \int_t^T Z(t,s)dW(s), \ \ 0 \leq t \leq T. \end{aligned} \tag{1.2}$$

When α is a Dirac measure on 0 i.e $\alpha = \delta_0$, BSVIEs (1.2) becomes

$$\begin{aligned} Y(t) &= F(t) + \int_t^T [G(t,s)Y(s) + g(t,s)Z(t,s)]ds \\ &\quad - \int_t^T Z(t,s)dW(s), \end{aligned} \tag{1.3}$$

which has been study in [13]. Authors provide under a suitable condition the so-called variation of constants formula for linear BSVIEs. More precisely, they provide the representation of Y as follows:

$$Y(t) = \mathbb{E}\left[F(t)M^t(T) + \int_t^T \Psi(t,r)M^r(T)F(r)dr|\mathcal{F}_t\right],$$

where for all $t \in [0,T]$, M^t is the solution to the following SDE:

$$dM^t(s) = g(t,s)M^t(s)dW(s), \quad t \leq s \leq T$$

and Ψ design a given function which will be specified later.

A similar result was first obtained by Hu and Oksendal, [9] for the linear BSVIEs driven by a Brownian motion and a compensated Poisson random measure. However, in [9] the coefficients G and g are assumed to be deterministic and the function g depends only on s (i.e $g(t,s) = g(s)$). In this context, BSVIE (1.3) becomes

$$\begin{aligned} Y(t) &= F(t) + \int_t^T [G(t,s)Y(s) + g(s)Z(t,s)]ds \\ &\quad - \int_t^T Z(t,s)dW(s), \end{aligned} \tag{1.4}$$

and Y defined by (1.4) can be represented as follows:

$$Y(t) = \mathbb{E}^{\mathbb{Q}}\left[F(t) + \int_t^T \Psi(t,r)F(r)dr|\mathcal{F}_t\right].$$

where $\mathbb{Q}$ is probability measure equivalent to $\mathbb{P}$. It is defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\int_0^T g(s)dW(s) - \frac{1}{2}\int_0^T g^2(s)ds\right).$$

In this paper we aim to present an explicit representation for the solution of linear delayed BSVIEs. Since previously we state that delayed linear BSVIEs becomes linear

BSVIEs when α, a probability measure is equal to δ_0 the Dirac measure, the general context is questionable. Therefore this work can be view as the general extension of one appear in [13] and then in [9], when we suppose G and g deterministic and the function g depends only of t (i.e $g(t,s)=g(s)$). Our method is essentially based on the transformation of delayed BSVIEs (1.2) to one without delay. Next with the similar argument used in [13] and [9] respectively, we establish an explicit expression for it solution.

Let describe two examples of applications which motivated our work.

First, we have its link with theory of hedging portfolios in a Volterra type delayed Black and Scholes market. Indeed, let consider a financial market made up of a risk-free asset and a risky asset described respectively by

$$S^0(t) = 1 + \int_0^t \int_{-T}^0 r(t+u, s+u) S^0(s+u)\alpha(u)ds$$

and

$$\begin{aligned} S(t) &= S_0 + \int_0^t \int_{-T}^0 \mu(t+u, s+u) S(s+u)\alpha(du)ds \\ &\quad + \int_0^t \sigma(t,s)S(s)dW(s), \quad S_0 > 0, \end{aligned}$$

where r, μ and σ are some functions and S_0 is a positive constant. Let $\theta = (\phi_0(t), \phi(t))$ be a portfolio of some operator and $Y(t)$ it value. Hence we have

$$\begin{aligned} Y(t) &= Y(0) + \int_0^t \int_{-T}^0 \phi^0(s) r(t+u, s+u) S^0(s+u)\alpha(u)ds \\ &\quad + \int_0^t \int_{-T}^0 \phi(s)\mu(t+u, s+u)S(s+u)\alpha(u)ds + \int_0^t \sigma(t,s)\phi(s)S(s)dW(s) \\ &= Y(0) + \int_0^t \int_{-T}^0 r(t+u, s+u)(Y(s+u) - \phi(s)S(t+u))\alpha(du)ds \\ &\quad + \int_0^t \int_{-T}^0 \phi(s)\mu(t+u, s+u)S(s+u)\alpha(u)ds + \int_0^t \sigma(t,s)\phi(s)S(s)dW(s) \\ &= Y(0) + \int_0^t r(t+u, s+u)Y(s+u)\alpha(du)ds \\ &\quad + \int_0^t \int_{-T}^0 (\mu(t+u, s+u) - r(t+u, s+u)\phi(s)S(t+u)ds + \int_0^t \sigma(t,s)\phi(s)S(s)dW(s). \end{aligned}$$

Setting $Z(t,s) = \sigma(t,s)\phi(s)S(s)$ and $\beta(t,s) = \dfrac{\mu(t,s) - r(t,s)}{\sigma(t,s)}$, the above equation becomes

$$\begin{aligned} Y(t) &= Y(0) + \int_0^t \int_{-T}^0 [r(t+u, s+u)\, Y(s+u) + \beta(t+u, s+u)\, Z(t+u, s+u)]\alpha(du)ds \\ &\quad + \int_0^t Z(t,s)dW(s). \end{aligned} \tag{1.5}$$

If we assume that this portfolio covers a playoff of some option with value $h(S(T))$ i.e $Y(T) = h(S(T))$, it follows from (1.5) that the portefeuille $(Y(t))_{t\geq 0}$ satisfies

$$\begin{aligned} Y(t) &= h(S(T)) - \int_t^T \int_{-T}^0 [r(t+u, s+u)\, Y(s+u) + \beta(t+u, s+u)\, Z(t+u, s+u)]\alpha(du)ds \\ &\quad - \int_t^T Z(t,s)\, dW(s). \end{aligned} \tag{1.6}$$

Since $Y(0)$ is the contract price at time 0, it is useful and even necessary to know its value.

The second example concerns delayed optimal consumption of a Volterra type cash flow introduced by Agram et al. in [1]. More precisely denote by $(X^u(t))_{t\geq 0}$, the cash flow modeled by the following stochastic Volterra equation with delay

$$\begin{cases} X(t) = x_0 + \int_0^t \int_{-\tau}^0 [b_0(t+v, s+v)X(s+v) - u(s+v)]\alpha(dv)ds + \int_0^t \sigma_0(s)X(s)dW(s), \quad t \geq 0, \\ X(t) = x_0,\ u(t) = \eta(t) \quad t \in [-\tau, 0]. \end{cases} \tag{1.7}$$

or, in differential form by

$$\begin{cases} dX(t) = \int_{-\tau}^0 [b_0(t+v, t+v)X(t+v) - u(t+v)]\alpha(dv)dt + \sigma_0(t)X(t)dW(t) \\ \qquad\qquad + \left[\int_0^t \int_{-\tau}^0 \frac{\partial b_0}{\partial t}(t+v, s+v)X(s+v)\alpha(dv)ds\right] dt, \quad t \geq 0, \\ X(t) = x_0,\ u(t) = \eta(t) \quad t \in [-\tau, 0]. \end{cases}$$

In this problem, we need to find a control u which maximizes the performance functional $J(u)$ defined by

$$J(u) = \mathbb{E}\left[\int_0^T \int_{-\tau}^0 \ln(u(s+v))\alpha(dv)ds + \int_{-\tau}^0 \theta(v)X^u(T+v)\alpha(dv)\right].$$

Roughly speaking, we need to find $\hat{u}$, such that

$$J(\hat{u}) = \sup_u J(u).$$

By using the maximum principle and in view of [4] or [1], the adjoints processes $(Y(t), Z(t,s))_{0\leq t\leq s\leq T}$ satisfy a backward stochastic Volterra type of the form:

$$Y(t) = \int_{-\tau}^0 \theta(v)\alpha(dv) + \int_t^T f(t, s, X_s, u_s, Y_s, Z_{t,s})ds - \int_0^T Z(t,s)dW(s), \tag{1.8}$$

where f is defined by

$$\begin{aligned} f(t, s, u_s, y, z) \quad &= \quad \int_{-\tau}^0 b_0(t+v, t+v)y(t+v)\alpha(dv) \\ &\quad + \sigma(t)z + \int_t^T \int_{-\tau}^0 \frac{\partial b_0}{\partial t}(t+v, s+v)y(s+v)\alpha(dv)ds, \end{aligned}$$

where x, u, y, z is defined in the appropriated space. Using the same methodology as one appear in [1], to derive $\hat{u}$, we need to solve BSVIE (1.8).

In view of his two examples, the study of the explicit solution of linear delayed BSVIEs comes very important especially since such study in not exists in the literature.

The rest of this note is organized as follows. In Section 2, we introduce some fundamental knowledge and assumptions concerning the data of BSVIEs (1.2). Section 3 is devoted to derive our result.

2. Preliminaries

In all this paper, we shall work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual condition. Let $W = (W(t),\ t \geq 0)$ be a standard one dimensional $\mathbf{F}$-adapted Brownian motion. To well define a notion of solution of BSVIEs with time delayed generator, let us consider the following spaces. For $\beta > 0$,

- : • Let $\mathbb{H}_1 := \mathcal{H}^2_{[-T,T]}(\mathbb{R})$ denote the space of all adapted process $(\eta(t))_{-T\le t\le T}$ with values in $\mathbb{R}$ such that $\eta(t)=\eta(0)$ for $t<0$ and
$$\|\eta\|^2_{\mathbb{H}_1} = \mathbb{E}\left[\left(\int_0^T e^{\beta s}|\eta(s)|^2 ds\right)^{1/2}\right] < +\infty.$$
- : • Let $\mathbb{H}_2 := \mathcal{H}^2_{D_T}(\mathbb{R})$ denote the space of all functions $\varphi : D_T \to \mathbb{R}$ such for all $t\in[-T,T]$, the process $\varphi(t,s)_{s\ge t}$ is adapted, $\varphi(t,s)=0$ if $t<0$ or $s<0$ and
$$\|\varphi\|^2_{\mathbb{H}_2} = \mathbb{E}\left[\left(\int_0^T\int_t^T e^{\beta s}|\varphi(t,s)|^2 dsdt\right)^{1/2}\right] < +\infty,$$
where $D_T = \{(t,s)\in[-T,T]^2, t\le s\}$.
- : • Let $\mathcal{S}^2(\mathbb{R})$ denote the space of all predictable and almost surely continuous process $(\eta(t))_{-T\le t\le T}$ with values in $\mathbb{R}$ such that the norm $\|\eta\|^2_{\mathcal{S}^2} = \mathbb{E}\left[\sup_{0\le s\le T} e^{\beta s}|\eta(s)|^2\right] < +\infty$.
- : • Let $L^2_{-T}(\mathbb{R})$ denote the space of measurable functions $z : [-T;0]\to\mathbb{R}$ satisfying
$$\int_{-T}^0 \mid z(t)\mid^2 dt < +\infty.$$
- : • Let $L^\infty_{-T}(\mathbb{R})$ denote the space of bounded, measurable functions $y : [-T,0]\to\mathbb{R}$ satisfying
$$\sup_{-T\le t\le 0} \mid y(t)\mid^2 < +\infty.$$

For the comprehension of the sequel, let us precise the following: for each $(t,s)\in D_T$, $(Y(t),Z(t,s))$ denotes the value of the process (Y,Z) at (t,s) while $(Y_t,Z_{t,s}) = (Y(t+u),Z(t+u,s+u))_{-T\le u\le 0}$ denotes all the past of (Y,Z) until (t,s). Therefore, for each $(t,s)\in D_T$ and almost all $\omega\in\Omega$, $Y_t(\omega)$ and $Z_{t,s}(\omega)$ belong respectively to $L^\infty_{-T}(\mathbb{R})$ and $L^2_{-T}(\mathbb{R})$.

Definition 2.1. *The process* $(Y(.),Z(.,.))$ *is called a solution to* (1.2) *if* $(Y(.),Z(.,.))$ *belongs to* $\mathbb{H}_1\times\mathbb{H}_2$ *and satisfies* (1.2).

According to Proposition 3.1 in [3], we have

Remark 1. *If* $(Y(.),Z(.,.))$ *belongs to* $\mathbb{H}^1\times\mathbb{H}^2$ *and satisfy* (1.2)*, then* $Y(.)\in\mathcal{S}^2(\mathbb{R})$.

Our result will be derived under the following assumptions.

- : **(A1)** $\{F(t), 0\le t\le T\}$ is a given stochastic process, not necessarily adapted that belongs in $\mathcal{S}^2(\mathbb{R})$.
- : **(A2)** $G : D_T\times\Omega\to\mathbb{R}$ is a progressively measurable, uniformly bounded function.
- : **(A3)** $g : D_T\times\Omega\to\mathbb{R}$ is a progressively measurable, uniformly bounded function.
- : **(A4)** The delay probability measure α have only one atom which is 0.

We give this remark in order to justify certain points of the preceding hypotheses.

Remark 2. *In order to extend* (Y,Z)*, the solution of* (1.2)*, to the time interval* $[-T,0]$ *such that* $Y(t)=Y(0)$ *and* $Z(t,s)=0$*, we extend the functions* F*,* G *and* g *respectively as follow: For all* $t<0$*,* $s<0$*,* $F(t)=F(0)$ *and* $G(t,s)=g(t,s)=0$.

3. Main result

We now establish the main result of the paper that consists to derive an explicit formula for solutions of BSVIE (1.2). We first give the following statement. Let us define
$$\Phi(t,r) = -\alpha(\{0\})G(t,r)$$

and consider the sequence $(\Phi^{(n)})_{n\geq 0}$ defined recursively as follows: $\Phi^{(1)}(t,r)=\Phi(t,r)$ and for all $n\geq 2$

$$\Phi^{(n)}(t,r)=\int_t^r \Phi^{(n-1)}(t,s)\Phi(s,r)ds.$$

We have

Remark 3. *Since α is a probability measure, there exists a constant $C>0$ (a uniform bound of the function G) such $|\Phi(t,r)|<C$. Moreover, by induction method we prove that for all $n\geq 2$,*

$$|\Phi^{(n)}(t,s)|\leq \frac{(CT)^n}{n!}.$$

Hence, for all t,s

$$\sum_{n=1}^{+\infty}|\Phi^{(n)}(t,s)|<+\infty.$$

In the sequel let us set

$$\Psi(t,r)=\sum_{n=1}^{\infty}\Phi^{(n)}(t,r)\quad\text{and}\quad \phi(t,r)=-\alpha(\{0\})g(t,r) \tag{3.1}$$

Theorem 3.1. *Assume that* (**A1**)-(**A4**) *hold. For a sufficiently small time horizon T or for a sufficiently small uniform bound C of G and g, i.e*

$$2C^2Te\max(1,T)<1, \tag{3.2}$$

BSVIE (1.2) *has a unique solution $(Y,Z)\in\mathcal{S}^2\times\mathbb{H}^2$.*

Moreover, this solution can be written by: for all $t\in[0,T]$,

(i)

$$Y(t)=\mathbb{E}\left[F(t)M^t(T)+\int_t^T\Psi(t,r)F(r)M^r(T)dr|\mathcal{F}_t\right];$$

(ii)

$$Z(t,s)=\mathbb{E}\left[M^t(s)\left(D_sU^t(t)-U(t)\int_s^T D_s\phi(t,r)dB^t(r)\right)|\mathcal{F}_s\right],\quad 0\leq t\leq s\leq T,$$

where

$$U(t)=F(t)+\int_t^T\Phi(t,r)Y(r)dr-Y(t), \tag{3.3}$$

with D_s and M^t denote respectively the Hida-Malliavin derivatives (see [9] *for more detail) and the exponential martingale solution of the SDE:*

$$dM^t(r)=M^t(r)\phi(t,r)dW(r),\ t\leq r\leq T;\quad M^t(t)=1.$$

Proof. Let us first prove that BSVIE (1.2) admits a unique solution. In this fact, thanks to the work of Coulibaly and Aman (see [3], Theorem 3.3), it suffices to show that its generator is delayed Lipschitz. For all $(t,s)\in D_T$ and any $(y_t,z_{t,s}),(y'_t,z'_{t,s})\in L^{\infty}_{-T}(\mathbb{R})\times L^2_{-T}(\mathbb{R})$, let set

$$f(t,s,y_s,z_{t,s})=\int_{-T}^0[G(t+u,s+u)y(s+u)+g(t+u,s+u)z(t+u,s+u)]\alpha(du).$$

hence

$$\begin{aligned}
& \left|f(t,s,y_s,z_{t,s})-f(t,s,y'_s,z'_{t,s})\right|^2 \\
\leq\ & 2\int_{-T}^{0} G^2(t+u,s+u)\alpha(du)\int_{-T}^{0}\left|y(s+u)-y'(s+u)\right|^2\alpha(du) \\
& +2\int_{-T}^{0} g^2(t+u,s+u)\alpha(du)\int_{-T}^{0}\left|z(t+u,s+u)-z'(t+u,s+u)\right|^2\alpha(du) \\
\leq\ & 2C^2\left(\int_{-T}^{0}\mid y(s+u)-y'(s+u)\mid^2\alpha(du)+\int_{-T}^{0}\mid z(t+u,s+u)-z'(t+u,s+u)\mid^2\alpha(du)\right).
\end{aligned}$$

Now it remain to prove (i) and (ii). Let us begin with (i). Our approach is as follows. First, we transform the BSVIE (1.2) into a BSVIE without delay. Then, we get (i) using [13]. Let us first integrate each term of BSVIE (1.2). We obtain for all $w \in (0,T)$,

$$\int_w^T Y(t)dt \quad = \quad H(w)+J(w)+I(w), \tag{3.4}$$

where

$$H(w)=\int_w^T F(t)dt,$$

$$J(w)=\int_w^T\int_t^T\int_{-T}^{0}[G(t+u,s+u)Y(s+u)+g(t+u,s+u)Z(t+u,s+u)]\alpha(du)dsdt$$

and

$$I(w)=-\int_w^T\left(\int_t^T Z(t,s)dW(s)\right).$$

On the other hand, setting

$$L(w)=\int_w^T Y(t)dt,$$

we have

$$L'(w)=H'(w)+J'(w)+I'(w). \tag{3.5}$$

According to their definition, we have

$$L'(w)=-Y(w), \tag{3.6}$$

$$H'(w)=-F(w) \tag{3.7}$$

and

$$I'(w)=\int_w^T Z(w,s)dW(s) \tag{3.8}$$

It remains for us to look very closely at $J'(w)$. Let us fist derive the explicit expression of $J(w)$. Applying respectively Fubini's theorem, change of variable and use the fact that

$G(t,s) = Z(t,s) = g(t,s) = 0$ for $t < 0$ or $s < 0$ we derive

$$\begin{aligned}
J(w) &= \int_{-T}^{0}\int_{w+u}^{T+u}\int_{t}^{T+u}(G(t,s)Y(s)+g(t,s)Z(t,s))dsdt\alpha(du)\\
&= \int_{-T}^{0}\int_{-T}^{T}\int_{t}^{T}\mathbf{1}_{[w+u,T+u]}(t)\mathbf{1}_{[t,T+u]}(s)(G(t,s)Y(s)+g(t,s)Z(t,s))dsdt\alpha(du)\\
&= \int_{0}^{T}\int_{t}^{T}\int_{-T}^{0}\mathbf{1}_{[s-T,(t-w)\wedge 0]}(u)\mathbf{1}_{[s-T,T]}(t-w)(G(t,s)Y(s)+g(t,s)Z(t,s))\alpha(du)dsdt\\
&= \int_{0}^{T}\int_{t}^{T}\int_{-T}^{0}\mathbf{1}_{[s-T,0]}(u)\mathbf{1}_{[s-T,T]}(t-w)\mathbf{1}_{[0,t]}(w)(G(t,s)Y(s)+g(t,s)Z(t,s))\alpha(du)dsdt\\
&\quad+\int_{0}^{T}\int_{t}^{T}\int_{-T}^{0}\mathbf{1}_{[s-T,(t-w)]}(u)\mathbf{1}_{[s-T,T]}(t-w)\mathbf{1}_{[t,T]}(w)(G(t,s)Y(s)+g(t,s)Z(t,s))\alpha(du)dsdt\\
&= \int_{0}^{T}\int_{t}^{T}\int_{s-T}^{0}\mathbf{1}_{[0,t]}(w)(G(t,s)Y(s)+g(t,s)Z(t,s))\alpha(du)dsdt\\
&\quad+\int_{0}^{T}\int_{t}^{T}\int_{s-T}^{0}\mathbf{1}_{[t,t-u]}(w)(G(t,s)Y(s)+g(t,s)Z(t,s))\alpha(du)dsdt\\
&= \int_{0}^{T}\int_{t}^{T}\int_{s-T}^{0}\mathbf{1}_{[0,t-u]}(w)(G(t,s)Y(s)+g(t,s)Z(t,s))\alpha(du)dsdt.
\end{aligned}$$

Hence,

$$\begin{aligned}
J'(w) &= \int_{0}^{T}\int_{t}^{T}\int_{s-T}^{0}(\delta_{t-u}(w)-\delta_0(w))(G(t,s)Y(s)+g(t,s)Z(t,s))\alpha(du)dsdt\\
&= \int_{0}^{T}\int_{t}^{T}\int_{T-s}^{0}\delta_{t-u}(w)(G(t,s)Y(s)+g(t,s)Z(t,s))\alpha(du)dsdt\\
&= \int_{0}^{T}\int_{t}^{T}\alpha(\{t-w\})(G(t,s)Y(s)+g(t,s)Z(t,s))dsdt.
\end{aligned}$$

Finally, according to (**A4**) we obtain

$$J'(w) = \int_{w}^{T}\alpha(\{0\})(G(w,s)Y(s)+g(w,s)Z(w,s))ds. \tag{3.9}$$

In view of (3.5)-(3.9), we have

$$\begin{aligned}
Y(t) &= F(t)+\int_{t}^{T}\Phi(t,s)Y(s)ds+\int_{t}^{T}\phi(t,s)Z(t,s)ds\\
&\quad-\int_{t}^{T}Z(t,s)dW(s), \quad a.s.
\end{aligned}\tag{3.10}$$

Then in view of Proposition 6.1 in [13], we have

$$Y(t) = \mathbb{E}\left(M^t(T)F(t)+\int_{t}^{T}\Psi(t,s)M^r(T)F(r)dr|\mathcal{F}_t\right),$$

where for all $t \geq 0,\ M^t$ is the solution to the following SDE

$$dM^t(s) = \phi(t,s)M^t(s)dW(s), \quad t \leq s \leq T, \quad M^t(t) = I.$$

Let us end this proof by provide (ii). For this, let fix $t \in [0,T]$ and define

$$V^t(s) = F(s)+\int_{s}^{T}\Phi(t,u)Y(u)du - Y(s), \quad s \geq t.$$

According to it definition and in view of (3.10), we have $V^t(t) = U(t)$ and

$$\begin{aligned} V^t(s) &= \int_s^T Z(t,u)dW(u) - \int_s^T \phi(t,u)Z(t,u)du \\ &= \int_s^T Z(t,u)dB^t(u), \quad t \le s, \end{aligned} \tag{3.11}$$

where for all $s \ge t$,

$$B^t(s) = W(s) - \int_0^s \phi(t,u)du,$$

In view of assumption on g, and for a fix $t \in [0,T]$, $(B^t(s))_{s\ge t}$ is a Brownian motion under the probability measure $\mathbb{Q}^t$ defined by $d\mathbb{Q}^t = M^t(T)d\mathbb{P}$, where the processus $(M^t(s))_{s\ge t}$ satisfies the following linear SDE

$$dM^t(s) = \phi(t,s)M^t(s)dW(s), \quad t \le s \le T, \quad M^t(t) = I. \tag{3.12}$$

On the other hand, for a fix $t \in [0,T]$, the process $(V^t(s))_{t\le s\le T}$ belongs to $L^2(\mathbb{Q}^t, \mathcal{F}_T)$. Next, according to the Clark-Ocone formula under change of measure (see [11]), extended to $L^2(\mathbb{Q}^t, \mathcal{F}_T)$ as in [12], we have, for all $s \in [t,T]$,

$$\begin{aligned} V^t(s) &= \mathbb{E}^{\mathbb{Q}^t}(V^t(s)) + \int_t^T \mathbb{E}^{\mathbb{Q}^t}\left[\left(D_v V^t(s) - V^t(s)\int_v^T D_v\phi(t,r)dB^t(r)\right)|\mathcal{F}_v\right]dB^t(v) \\ &= \mathbb{E}^{\mathbb{Q}^t}[V^t(s)] + \int_t^T \mathbb{E}^{\mathbb{Q}^t}\left[\left(D_v V^t(s) - V^t(s)\int_v^T D_v\phi(t,r)dB^t(r)\right)|\mathcal{F}_v\right]dB^t(v). \end{aligned}$$

In particular for $s = t$, we get

$$U(t) = \mathbb{E}^{\mathbb{Q}^t}(U(t)) + \int_t^T \mathbb{E}^{\mathbb{Q}^t}\left[\left(D_v U(t) - U(t)\int_v^T D_v\phi(t,r)dB^t(r)\right)|\mathcal{F}_v\right]dB^t(v).$$

Finally, since $\mathbb{E}^{\mathbb{Q}^t}(U(t)) = 0$, it follows from identification with the equality (3.11) that

$$\begin{aligned} Z(t,s) &= \mathbb{E}^{\mathbb{Q}^t}\left[\left(D_s U^t(t) - U(t)\int_s^T D_s\phi(t,r)dB^t(r)\right)|\mathcal{F}_s\right] \\ &= \mathbb{E}\left[M^t(s)\left(D_s U^t(t) - U(t)\int_s^T D_s\phi(t,r)dB^t(r)\right)|\mathcal{F}_s\right], \quad 0 \le t \le s \le T, \end{aligned}$$

where $(M^t(s))_{s\ge t}$ satisfies (3.12). □

Now, if we suppose function g defined only on $[-T,T]$, we have the following:

Corollary 3.3. *Assume that* (**A1**)-(**A3**) *hold. If the horizon time T or bound of G and g are small enough, then BSVIE* (1.2) *has a unique solution $(Y,Z) \in \mathcal{S}^2 \times \mathbb{H}^2$. Moreover, we have*

(i)

$$Y(t) = \mathbb{E}^{\mathbb{Q}}\left[F(t) + \int_t^T \Psi(t,r)F(r)dr|\mathcal{F}_t\right],$$

where $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \alpha(\{0\})g(s)dW(s) - \frac{1}{2}\int_0^T \alpha^2(\{0\})g^2(s)ds\right). \tag{3.13}$$

(ii) *For all $0 \le t \le T$, we have*

$$Z(t,s) = \mathbb{E}^{\mathbb{Q}}\left[D_s U(t) - U(t)\int_s^T D_s g(r)dW^{\mathbb{Q}}(r)|\mathcal{F}_s\right]; \quad 0 \le t \le s \le T,$$

where D_s denotes the Hida-Malliavin derivatives state briefly in [9].

Let us illustre our result by an adapted version of example appear in [9].

Example 1. *As in* [9]*, let* ρ *be a bounded function defined on* $[0,+\infty[$*,* $\mathcal{L}\rho$ *it Laplace transform defined by*

$$\mathcal{L}\rho(x) = \int_0^{+\infty} e^{-xy}\rho(y)dy$$

and α *a probability measure on* $[-T,0]$ *defined by* $\alpha = \frac{1}{2T}\lambda + \frac{1}{2}\delta_0$*, where* λ *is Lebesgue measure. Let set*

$$G(t,s) = -2(s-t)\rho(s-t), \quad 0 \le t \le s \le T.$$

Hence, we have

$$\begin{aligned} \Phi(t,s) &= -\alpha(\{0\})G(t,s) \\ &= (s-t)\rho(s-t) = \psi(s-t). \end{aligned}$$

Thus, $\Phi^{(n)}(t,s) = \psi_n(t-s)$ *and*

$$\Psi(t,s) = \sum_{n=0}^{+\infty} \psi_n(t-s),$$

where ψ_n *denotes the* n *fold convolution of* ψ*. Since we have*

$$\mathcal{L}\left(\sum_{n=1}^{+\infty}\psi_n\right)(x) = \sum_{n=1}^{+\infty}\mathcal{L}\psi_n(x),$$

and

$$\mathcal{L}\psi_n(x) = (\mathcal{L}\psi(x))^n$$

we derive

$$\begin{aligned} \mathcal{L}\Psi(t,s) &= \sum_{n=1}^{+\infty}(\mathcal{L}\psi(s-t))^n \\ &= \frac{\mathcal{L}\psi(s-t)}{1-\mathcal{L}\psi(s-t)}. \end{aligned}$$

where in last equality, we have assumed that $|\mathcal{L}\psi(x)| < 1$*, for all* $x > 0$*. Specially taking* $\rho(y) = e^{-y}, \ y > 0$*, we provide that*

$$\mathcal{L}\psi(x) = \frac{1}{(x+1)^2}. \tag{3.14}$$

Set $\Psi(t,s) = \overline{\psi}(s-t)$*, it follows from* (3.14) *that*

$$\begin{aligned} \mathcal{L}\overline{\psi}(x) &= \sum_{n=1}^{+\infty}(\mathcal{L}\psi(x))^n \\ &= \frac{\mathcal{L}\psi(x)}{1-\mathcal{L}\psi(x)} \\ &= \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+2}\right). \end{aligned}$$

Finally, by converse Laplace transform, we derive

$$\Psi(t,s) = \frac{1}{2}(1-e^{-(s-t)}).$$

and

$$Y(t) = \mathbb{E}^{\mathbb{Q}}\left(F(t) + \frac{1}{2}\int_t^T (1-e^{-(s-t)})F(s)ds|\mathcal{F}_t\right).$$

The smoothness properties of $Z(t,s)$ with respect to t are important in the study of optimal control (see, [1]) and the numerical solutions (see, [8] and references therein). Using the explicit form of the solution (Theorem 3.3) we can give sufficient conditions for such smoothness in the linear case.

Theorem 3.2. *Suppose g be deterministic, the function F and G are almost surely $\mathcal{C}^1$ with respect to variable t and satisfy*

$$\mathbb{E}\left[\int_0^T \left\{ M^t(T)\int_t^T \left(F^2(t) + \alpha^2(\{0\})G^2(t,s) + (F'(t))^2 + \alpha^2(\{0\})\left(\frac{\partial G(t,s)}{\partial t}\right)^2\right) ds \right\} dt\right] < +\infty.$$

Then for $t < s \leq T$,

$$Z(t,s) = \mathbb{E}\left[M^t(s)D_sF(t) - \int_s^T M^t(s)\alpha(\{0\})G(t,r)D_sY(r)dr|\mathcal{F}_s\right]. \tag{3.15}$$

Moreover, we get

$$\mathbb{E}\left[\int_0^T \left\{ M^t(T)\int_t^T \left(\frac{\partial Z(t,s)}{\partial t}\right)^2 ds\right\} dt\right] < +\infty. \tag{3.16}$$

Proof. It follows from (ii) of Theorem 3.2 that

$$\begin{aligned} Z(t,s) &= \mathbb{E}\left[M^t(s)D_sU^t(t)|\mathcal{F}_s\right] \\ &\quad -\mathbb{E}\left[M^t(s)U(t)\int_s^T D_s\phi(t,r)dB^t(r)|\mathcal{F}_s\right]. \end{aligned}$$

But, since $M^t(s)$ is $\mathcal{F}_s$-mesurable and using the same argument appear in [9], we obtain

$$\begin{aligned} \mathbb{E}\left[M^t(s)U(t)\int_s^T D_s\phi(t,r)dB^t(r)|\mathcal{F}_s\right] &= M^t(s)\mathbb{E}\left[U(t)\int_s^T D_s\phi(t,r)dB^t(r)|\mathcal{F}_s\right] \\ &= 0. \end{aligned}$$

Therefore in view of (3.3) and the fact that $D_sY(t) = 0$, for $s > t$, we get

$$\begin{aligned} Z(t,s) &= \mathbb{E}\left[M^t(s)D_sU^t(t)|\mathcal{F}_s\right] \\ &= \mathbb{E}\left(M^t(s)D_sF(t) - \int_s^T M^t(s)\alpha(\{0\})G(t,r)D_sY(r)dr|\mathcal{F}_s\right). \end{aligned}$$

Hence (3.18) holds. □

Now, if we suppose function g defined only on $[-T,T]$, we have the following:

Corollary 3.3. *Assume the same assumptions of Theorem 3.2. For $t < s \leq T$, we have*

$$Z(t,s) = \mathbb{E}^{\mathbb{Q}}\left[D_sF(t) - \int_s^T \alpha(\{0\})G(t,r)D_sY(r)dr|\mathcal{F}_s\right]. \tag{3.17}$$

Moreover, we get

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \left\{ \int_t^T \left(\frac{\partial Z(t,s)}{\partial t}\right)^2 ds\right\} dt\right] < +\infty, \tag{3.18}$$

where $\mathbb{Q}$ is defined by (3.13).

Remark 4. *In view of all above, we can say that when we suppose that the delay probability measure defined on $[-T,0]$ has only 0 as a unique atom, the delayed linear BSVIEs can be seen as a generalization of the classical linear BSVIEs. Indeed, in view of the proof of Theorem 3.3, we have transformed BSVIEs* (1.2) *to classical linear BSVIE with generator depending to a delayed probability measure.*

Conflict of interest statement. We declare that there is no conflict of interest associated with this work. We have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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