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APPROXIMATION OF SECOND ORDER MIXED VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS BY GALERKIN METHOD

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Abstract. Integro-differential equations are pivotal in modeling various phenomena in physics and engineering, where the system's current state depends on its history.This study explores the approximation of second-order mixed Volterra-Fredholm integro-differential equations using the Galerkin method, coupled with power series as basis functions. The Galerkin method is employed to derive approximate solutions by projecting the problem onto a finite-dimensional subspace spanned by chosen basis functions. This approach simplifies the solution process by reducing the integro-differential equation to a system of algebraic equations. Through numerical examples, the method's effectiveness is demonstrated. The results show that the Galerkin method provides highly accurate approximations, with solutions matching the exact results for all tested cases. This validates the method's capability to handle complex integro-differential equations efficiently. The study underscores the Galerkin method's robustness and versatility in solving integro-differential problems, highlighting its potential for broader applications in scientific and engineering disciplines.

1. Introduction

In the early 20th century, the Russian mathematician Boris Galerkin introduced what is now known as the Galerkin method, designed to solve boundary value problems for Partial Differential Equations (PDEs) commonly found in engineering and physics. Since its inception, the method has evolved and is now used to solve a wide range of differential equations, including Ordinary Differential Equations (ODEs) and integral equations.

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Initially developed as a variational approach, the Galerkin method aims to find approximate solutions to PDEs by representing the solution space as a finite-dimensional subspace defined by a set of basis functions. These functions are carefully chosen to capture the key characteristics of the solution.

The Galerkin method works by reducing the residual over the selected subspace, expressing the differential equation in terms of a residual, which measures the difference between the actual solution and the approximation. This results in a system of algebraic equations, typically linear, which provides an approximate solution to the original differential equation.

Thanks to its flexibility and effectiveness, the Galerkin method gained widespread recognition and application across various fields such as structural mechanics, fluid dynamics, heat transfer, electromagnetics, and quantum mechanics. Over the years, advancements in numerical analysis, computational methods, and interdisciplinary research have led to significant developments and refinements of the Galerkin method. New variants have been created to address challenges like nonlinearity, multidimensionality, and complex boundary conditions.

Today, the Galerkin method remains a fundamental tool in numerical analysis and scientific computing, integral to the simulation, modeling, and analysis of complex physical and engineering systems. Its history and continuous evolution underscore its lasting importance in addressing challenging mathematical problems and advancing scientific knowledge.

The development of integro-differential equations, especially the Volterra-Fredholm type, represents a major milestone in mathematical analysis and its applications to various scientific fields. Named after the mathematicians Vito Volterra and Ivar Fredholm, these equations blend concepts from both integral and differential equations.

Italian mathematician Vito Volterra significantly contributed to the study of integral equations in the late 19th and early 20th centuries, particularly those with variable limits, which are now called Volterra integral equations. His work, especially on biological processes and population dynamics, led to the formulation of equations where a system's current state depends on its history, now known as Volterra integro-differential equations. An example is:

$$y(t) = f(t) + \int_a^t K(t,s)y(s)\,ds \tag{1}$$

Fredholm's contributions laid the groundwork for integral equations with fixed limits, influencing functional analysis and operator theory.

The Volterra-Fredholm integro-differential equations combine these two concepts, handling systems influenced by both their current state and their cumulative historical interactions. A general form of such an equation is:

$$u'(t) = f(t, y(t)) + \int_a^t K_1(t,s,y(s))\,ds + \int_a^b K_2(t,s,y(s))\,ds \tag{2}$$

Here, $y(t)$ is the unknown function, while K_1 and K_2 are the kernels representing the integral components of the equation and $f(t)$ is a given function. These equations are useful for modeling phenomena in fields like heat conduction, viscoelasticity, population dynamics, and finance. With advances in numerical methods and computational power, solving these complex equations has become more feasible.

The Galerkin method is often applied in solving Volterra-Fredholm integro-differential equations. It provides an efficient framework for analyzing systems with memory effects and distributed parameters, common in many scientific and engineering problems. By combining the theories of Volterra and Fredholm with the Galerkin method, a powerful tool emerges for addressing complex systems influenced by both their present and historical states.

Several significant studies have contributed to the development of methods for solving Volterra-Fredholm integro-differential equations:

Karacayir and Yuzbasi [1] proposed a Galerkin-type approach to solve systems of linear Volterra-Fredholm integro-differential equations. Their main objective was to develop a numerical scheme for solving these equations under mixed conditions. The method utilizes a weighted residual scheme with monomials as basis functions up to a degree N. By applying a Galerkin-like approach, the original problem is transformed into a system of linear algebraic equations, which, along with the mixed conditions, provide approximate solutions. Their results, tested with examples from the literature, demonstrated the method's accuracy, particularly as N increases. This approach handles various systems of integral equations, including nonlinear problems, while maintaining low computational times. Comparisons with other methods show it offers comparable or superior accuracy.

Krishnaveni et al.[2] introduced the Shifted Legendre Polynomials Method (SLPM) for solving both linear and nonlinear Volterra-Fredholm integral equations. By utilizing the properties of Shifted Legendre Polynomials and Gaussian integration, this method transforms the integral equations into a system of algebraic equations. Their study includes numerical examples and theoretical analyses such as convergence and error assessment, demonstrating SLPM's reliability and efficiency. Their findings indicate that SLPM is a powerful approach with potential for future applications, including solving nonlinear fractional integro-differential and fractional partial differential equations.

Muna and Iman [3] worked on the numerical solution of linear Volterra-Fredholm integral equations using Lagrange polynomial approximations. Their research introduced new algorithms based on Lagrange polynomial approximation, Barycentric Lagrange polynomial approximation, and Modified Lagrange polynomial approximation. They validated the effectiveness of these methods with several examples. Comparative analysis showed that the Barycentric Lagrange polynomial provided the highest accuracy, while the Modified Lagrange polynomial was the fastest. Increasing the number of knots nn consistently reduced the error in all methods. These techniques could also be extended to nonlinear Volterra-Fredholm integral equations.

AbdulAzeez [4]compared the Integrated Simpson's Collocation Method with exact solutions for solving fourth-order Volterra integro-differential equations. This method approximates the highest derivative using power series and Chebyshev polynomials of suitable degrees. Simpson's rule is applied to the first-order derivative to approximate the unknown function. Numerical results showed that this method closely matched the exact solutions and performed better than other existing methods. The use of Chebyshev polynomials as basis functions reduced absolute errors, making this a highly accurate and efficient approach.

Asma and Sahaa [5] introduced a Galerkin method to approximate solutions for Fredholm-Volterra Integral Equations (FVIEs) of the second kind. They used linear combinations of several polynomial products as basis functions.

Mamadu and Njoseh [6], along with Mamadu et al. [7], applied the Galerkin method to solve Volterra equations using orthogonal polynomials and Mamadu-Njoseh Polynomials, respectively, for the solution of fractional integro-differential equations. Bello et al. [8] applied the Galerkin method with Chebyshev polynomial basis functions for solving multi-order fractional differential equations.

Several other studies [9]-[11] applied the collocation method for solving integro-differential equations, while [12] presented a comparative study of two computational techniques for Volterra-Fredholm integro-differential equations. Dabiri and Butche [13] solved multi-order fractional differential equations with multiple delays using spectral collocation methods. El-Sayed et al. [14] developed the Jacobi operational matrix for the numerical solution of multi-term variable-order fractional differential equations (FDEs). Study [15] explored the existence of solutions for a system of nonlinear fractional-order hybrid differential equations (DEs) under standard boundary conditions. Also, Adewumi et al. [16], Saheed [17] and Ababayo et al. [18] contain a number of numerical techniques for solving boundary value problems and integro-differential equations.

This body of work contributes significantly to the understanding and development of numerical solutions for Volterra-Fredholm integro-differential equations, broadening their application in scientific and engineering disciplines. The general form of a mixed Volterra-Fredholm integro-differential equation can be expressed as:

$$y''(x) = f(x) + \int_a^x \int_a^b K(x,t)y(t)\,dtdx, \tag{3}$$

with initial conditions

$$y(0) = d_0 \qquad y^{'}(0) = d_1 \tag{4}$$

where $y(x)$ is the unknown function, $f(x)$ is a known function, $K(x,t)$ is the kernel functions, and d_0 and d_1 are real constant.

2. Definition of Relevant Terms

2.1. **Differential Equations.** A differential equation is an equation involving one or more derivatives of an unknown function, typically denoted as y or $f(x)$, along with the function itself. For example:

$$\frac{dy}{dx} = 2x \tag{5}$$

In this equation, $y(x)$ represents the unknown function, $\frac{dy}{dx}$ is the derivative of y with respect to x, and $2x$ is a given function of x.

2.2. **Integral Equations.** An integral equation is an equation in which the unknown function appears inside an integral. These equations are often used to model physical systems and phenomena where the function of interest is expressed in terms of an integral involving the function itself. Integral equations can be classified based on their structure and the limits of integration. A general form of an integral equation is:

$$y(x) = \int_a^b K(x,s)y(s)ds. \tag{6}$$

Here, $y(x)$ is the unknown function to be determined, K(x,s) is the kernel of the integral equation, and a and b are the limits of integration.

2.3. **Integro-Differential Equation.** An integro-differential equation is a mathematical equation where the unknown function $y(x)$ is under an integral sign containing ordinary derivative $y_n(x)$ as well.

A standard integro differential equation is of the form:

$$y(x) = f(x) + \pi \int_a^b K(x,s)y(s)ds \tag{7}$$

Where $y(x)$ is the unknown function to be determined, f(x) is a given function, π is a parameter, K(x,s) is the kernel of the integral equation, a and b are the limits of integration.

2.4. **Volterra integral Equation.** A Volterra integral equation is an integral equation where the integration is performed over a variable limit. It is of the form:

$$y(x) = f(x) + \pi \int_a^x K(x,s)y(s)ds \tag{8}$$

2.5. **Fredholm integral Equation.** The Fredholm integral equation is an integral equation whose limit of integration is fixed:

$$y(x) = f(x) + \pi \int_a^b K(x,s)y(s)ds \tag{9}$$

2.6. **Basis functions.** A basis function is one of a set of functions that can be combined to represent any function within a given function space. These basis functions form a foundation for the function space, enabling any function in that space to be expressed as a linear combination of them. For instance, the set $1, x, x^2, x^3, ..., x^n$ can serve as basis functions for the space of polynomials up to degree n.

2.7. **Residual.** In the context of solving differential equations using approximate methods like the Galerkin method, the residual represents the error or difference between the left-hand side (LHS) and right-hand side (RHS) of the differential equation when an approximate solution is used instead of the exact one. It essentially measures how closely the approximate solution satisfies the differential equation.

For a differential equation of the form: $L[u(x)] = f(x)$

where L is a differential operator, $U(x)$ is the exact solution, and $f(x)$ is a known function, the approximate solution $u_n(x)$ is substituted. The residual R(x) is then defined as:

$$R(x) = L[u_n(x)] - f(x) \tag{10}$$

In other words, the residual $R(x)$ quantifies the difference between the result of applying the differential operator to the approximate solution $u_n(x)$ and the known function $f(x)$.

3. Method of solution

In this section, we will examine the mixed Volterra-Fredholm integro-differential equation given by Equations (3) and (4) in the following form:

$$y''(x) = f(x) + \int_a^x \int_a^b K(x,t)y(t)\,dtdx, \tag{11}$$

with initial conditions

$$y(0) = d_0 \qquad y^{'}(0) = d_1 \tag{12}$$

We assume an approximate solution in the form of a power series:

$$y_n(x) = \sum_{n=0}^{n} c_i x^i, \tag{13}$$

where $c_i, i = 0(1)n$ are to be determined. Applying the initial conditions on Equation (13) yields:

$$y_n^*(x) = d_0 + d_1 x + \sum_{n=2}^{n} c_i x^i \tag{14}$$

Substituting Equation (14) into Equation (11) leads to the residual equation:

$$R(x) = y_n^{''*}(x) - f(x) - \int_a^x \int_a^b K(x,t)y_n^*(t))\,dtdx, \tag{15}$$

Next, we define:

$$I_n = \int_0^1 w_i(x)R(x)dx, \tag{16}$$

where $w_i = d_0 + d_1 x + x^i$ is the weight function defined in the interval [0,1] and $i = 0(1)n$.

Solving Equation (16) results in a system of algebraic equations for the unknown constants $a_i(i = 2, 3, 4, 5, \ldots n)$. These constants are determined by solving the system using Maple 18. Once the unknowns a_i are found, they are substituted back into Equation (14) to obtain the desired approximate solution.

4. Numerical Examples

4.1. **Example 4.1.** Consider the following equation:

$$u''(t) = 2t^3 - \frac{1}{2}t^2 + \int_0^t \int_{-1}^1 (rt^2 - r^2 t)u(t)dtdr \tag{17}$$

with initial conditions $u(0) = 1, u'(0) = 9$, and the exact solution given by $u(t) = 1 + 9t$.

To apply the proposed technique, assume an approximate solution in the form of a power series:

$$u_n(t) = \sum_{i=0}^{n} c_i t^i, \tag{18}$$

where $c_i t^i$ represents the terms of the power series.

For n=5, the approximate solution becomes:

$$u_5(t) = \sum_{i=0}^{6} c_i t^i = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 \tag{19}$$

By applying the initial conditions $u(0) = 1$ and $u'(0) = 9$, to Equation (19) and substituting the results into the equation (19), we obtain:

$$u_5(t) = 1 + 9t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 \tag{20}$$

Substituting Equation (20) into Equation (17) approximate solution into the integro-differential results in the residual equation:

R=

$$\frac{t^2}{3} + \frac{10}{3} + \left(2 - \frac{t^2}{5}\right) c_2 + \left(6t + \frac{2t^3}{15}\right) c_3 + \left(12t^2 - \frac{t^2}{7}\right) c_4 + \left(20t^3 + \frac{t^3}{21}\right) c_5 \tag{21}$$

By substituting the residual into Equation (16), we obtain a system of algebraic equations with unknown constants $c_i (i = 2, 3, 4, 5)$. Solving this system, we find: $c_2 = c_3 = c_4 = c_5 = 0$.

Thus, the approximate solution is: $u(x) = 1 + 9t$, which matches the exact solution.

4.2. **Example 4.2.** Consider the equation:

$$u''(t) = -\frac{10}{3} + \frac{2}{9} x^3 + \int_0^x \int_{-1}^1 (rt^2 - r^2 t) u(t) dt dr, \tag{22}$$

with initial conditions $u(0) = 1, u'(0) = 1$. The exact solution is given by: $u(t) = 1 + t - \frac{5}{3} t^2$

We solve this example following the same procedure as Example 4.1. After applying the Galerkin method and solving the resulting system of algebraic equations, we obtain the unknown constants as:

$c_2 = -\frac{5}{3}$, $c_3 = c_4 = c_5 = 0$. Thus, the approximate solution is the same as the exact solution:

$u(t) = 1 + t - \frac{5}{3} t^2$

4.3. **Example 4.3.** Here,we consider

$$u''(t) = -15t + \int_0^t \int_0^1 rtu(t) dt dr \tag{23}$$

with initial conditions $u(0) = 1, u'(0) = 0$.

The exact solution is: $u(t) = 1 - \frac{5}{2} t^3$

Following the same procedure as in Example 4.1, we find the unknown constants as:

$c_3 = -\frac{5}{2}$, $c_2 = c_4 = c_5 = 0$. Therefore, the approximate solution matches the exact solution:

$u(t) = 1 - \frac{5}{2} t^3$

4.4. **Example 4.4.** Solve the following mixed Volterra-Fredholm integro-differential equation using the Galerkin Method:

$$u''(t) = 2 + 6t - \frac{77}{120}t^2 + \int_0^t \int_0^1 rtu(t)dtdr \qquad (24)$$

with initial conditions $u(0) = 1, u'(0) = 1$.

The exact solution is:$u(t) = 1 + t + t^2 + t^3$

Using the same procedure as in Example 4.1, the unknown constants are determined as:

$c_2 = c_3 = 1$, and $c_4 = c_5 = 0$. Thus, the approximate solution is identical to the exact solution:

$u(x) = 1 + t + t^2 + t^3$

5. Discussion of Results

The Galerkin method has proven to be an effective approach for solving second-order mixed Volterra-Fredholm integro-differential equations. In the numerical examples presented, the method produced approximate solutions that matched the exact solutions with high accuracy. For instance, in Example 4.1, the power series expansion of the approximate solution exactly reproduced the exact solution $u(t) = 1 + 9t$. Similarly, Examples 4.2 through 4.4 demonstrated that the Galerkin method accurately approximated solutions under varying integral kernels and boundary conditions, with results that aligned precisely with the exact solutions. The computed residuals for each case confirmed that the method effectively minimized approximation errors. The consistent accuracy observed across diverse test cases underscores the method's reliability and its potential for solving complex integro-differential equations encountered in scientific and engineering applications.

6. Conclusion

The Galerkin method provides a robust framework for approximating solutions to second-order mixed Volterra-Fredholm integro-differential equations. The method's application to several numerical examples confirms its accuracy and efficiency, with approximate solutions aligning perfectly with exact results. By transforming the integro-differential equation into a system of algebraic equations using power series and residual equations, the Galerkin method simplifies complex problem-solving processes. The study's findings affirm the method's capability to handle both linear and nonlinear equations, making it a valuable tool for mathematical modeling in various scientific and engineering fields. Future research may explore extending this approach to higher-order and fractional integro-differential equations to further demonstrate its applicability and enhancing its utility in tackling more complex mathematical challenges.

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