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$*$-WEYL CURVATURE TENSOR WITH IN FRAME WORK ON SASAKIAN MANIFOLD ADMITTING ZAMKOVOY CONNECTION

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Abstract. In this paper, we investigate the $*$-Weyl curvature tensor on Sasakian manifolds equipped with the Zamkovoy connection. We explore the geometric properties of $*$-Weyl flat and ξ-$*$-Weyl flat Sasakian manifolds under this connection. Furthermore, we examine the condition $\overline{W}^*(\xi, U) \circ \overline{R} = 0$, where $\overline{W}^*$ and $\overline{R}$ denote the $*$-Weyl curvature tensor and the Riemannian curvature tensor, respectively, both defined with respect to the Zamkovoy connection. To illustrate the theoretical results, we present an explicit example of a three-dimensional Sasakian manifold.

1. Introduction

The notion of a Sasakian structure [14] was introduced by the Japanese mathematician S. Sasaki in 1960. A contact metric manifold is said to be Sasakian if it is normal. In certain respects, Sasakian manifolds can be regarded as odd-dimensional analogues of Kähler manifolds. Weyl [12, 13] introduced a generalized curvature tensor on a Riemannian manifold that vanishes whenever the metric is (locally) conformally equivalent to a flat metric. This tensor, known as the Weyl conformal curvature tensor, is defined by

$$\begin{aligned} W(X,Y)Z &= R(X,Y)Z - \frac{1}{(2n-1)}\{Ric(Y,Z)X - Ric(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\} + \frac{1}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{1}$$

for all X, Y, $Z \in \chi(M)$, with R being the Riemannian curvature tensor, Ric being the Ricci tensor and r being the scalar curvature of M. If $n = 3$ then $W(X,Y)Z = 0$ and if $n \geq 4$ then M is locally conformal flat if and only if $W(X,Y)Z = 0$. In

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[9], the authors classified a classes of conformally flat contact metric manifolds and characterized a conformally flat contact manifolds as a hypersurfaces of 4-dimensional Kaehler Einstein manifolds. Miyazawa and Yamaguchi [11] proved that a conformally symmetric Sasakian manifold is also locally isometric to the unit sphere.

In 1959 Tachibana [16] defined $*$-Ricci tensor Ric^* on almost Hermitian manifold. In [5] Hamada gave the definition of $*$-Ricci tensor Ric^* in the following way

$$Ric^*(X,Y) = \frac{1}{2} trace(Z \to R(X, \phi Y)\phi Z),$$

for all vector fields $X, Y \in \chi(M)$, Weyl also introduced the notion of $*$-Einstein manifolds, characterized by the condition $g(Q^*X, Y) = \lambda g(X, Y)$, where Q^* is the $*$-Ricci operator and λ is a scalar function on the manifold. He further provided a classification of $*$-Einstein hypersurfaces. Ivey and Ryan [7] extended Hamada's work by examining the equivalence of the $*$-Einstein condition with other geometric conditions, such as the pseudo-Einstein and pseudo-Ryan conditions. Using the concept of the $*$-Ricci tensor, Aruna, Venkatesh, and Naik [6] investigated certain curvature properties on contact metric generalized (κ, μ)-space forms. Additionally, the authors in [18] explored curvature properties of Kenmotsu manifolds using the $*$-Ricci tensor. Unal, Sari, and Prakasha [17] studied the $*$-Ricci tensor on normal metric contact pair manifolds, while in [1], the $*$-Ricci tensor was examined in the context of α-cosymplectic manifolds. More recently, Kaimakamis and Panagiotidou [8] introduced the concept of the $*$-Weyl curvature tensor on real hypersurfaces in non-flat complex space forms. It is defined as follows:

$$\begin{aligned} W^*(X,Y)Z &= R(X,Y)Z - \frac{1}{(2n-1)}\{Ric^*(Y,Z)X - Ric^*(X,Z)Y + g(Y,Z)Q^*X \\ &- g(X,Z)Q^*Y\} + \frac{r^*}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned}$$

For all $X, Y, Z \in \chi(M)$, where Q^* is the $*$-Ricci operator and r^* is the $*$-scalar curvature corresponding to Q^* on M.

In 2008, the notion of the Zamkovoy connection was introduced by Zamkovoy [19] for paracontact manifolds. This connection is defined as a canonical paracontact connection whose torsion represents the obstruction for a paracontact manifold to be para-Sasakian. Let M be an n-dimensional almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$-tensor field, ξ is a vector field, η is a 1-form, and g is a Riemannian metric. The Zamkovoy connection $\overline{\nabla}$ is defined by

$$\overline{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)Y - \eta(Y)\nabla_X \xi + \eta(X)\phi Y \tag{2}$$

for all $X, Y \in \chi(M)$, this connection was further studied by A.M. Blaga [2] in the context of para-Kenmotsu manifolds. In a Sasakian manifold M of dimension $(2n+1)$, the $\overline{W^*}$ with respect to the Zamkovoy connection is given by

$$\begin{aligned} \overline{W^*}(X,Y)Z &= \overline{R}(X,Y)Z - \frac{1}{(2n-1)}\{\overline{Ric^*}(Y,Z)X - \overline{Ric^*}(X,Z)Y \\ &+ g(Y,Z)\overline{Q^*}X - g(X,Z)\overline{Q^*}Y\} + \frac{r^*}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{3}$$

where $\overline{R}, \overline{Ric}$ and $\overline{Q}$ are Riemannian curvature tensor, Ricci tensor and Ricci Operator with respect to Zamkovoy $\overline{\nabla}$ connection respectively.

2. **Preliminaries**

Let M be an almost contact metric manifold of dimension $(2n+1)$, equipped with an almost contact metric structure (ϕ,ξ,η,g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form, and g is a Riemannian metric. Then, according to [3], we have:

$$\phi^2 = -I + \eta\otimes\xi, \;\; \eta(\xi) = 1, \;\; \phi\xi = 0, \;\; \eta o\phi = 0, \tag{4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \tag{5}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{6}$$

for all vector fields X, Y on M.
A normal contact metric manifold is called a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if [15]

$$(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{7}$$

for all vector fields X, Y on M, where ∇ is the Levi-Civita connection associated with the Riemannian metric g. From the above equation, it follows that

$$\nabla_X\xi = -\phi X, \tag{8}$$

$$(\nabla_X\eta)Y = -g(\phi X, Y), \tag{9}$$

Moreover, the curvature tensor R, the Ricci tensor Ric and the Ricci operator Q in a Sasakian manifold M with respect to the Levi-Civita connection satisfy:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{10}$$

$$Ric(X, \xi) = 2n\eta(X), Q\xi = 2n\xi, \tag{11}$$

$$Ric(\phi X, \phi Y) = Ric(X, Y) - 2n\eta(X)\eta(Y), \tag{12}$$

$$\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y). \tag{13}$$

for any vector elds X, Y and Z on M .
In [4], Ghosh and Patra derived the expression of ∗-Ricci tensor on Sasakian manifold, which is of the form

$$Ric^*(X, Y) = Ric(X, Y) - (2n-1)g(X, Y) - \eta(X)\eta(Y), \tag{14}$$

$$r^* = r - 4n^2. \tag{15}$$

Definition 2.1. *A Sasakian manifold M is said to be an η-Einstein manifold if its Ricci tensor Ric is of the form*

$$Ric(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{16}$$

where a and b are scalar functions on M.

Definition 2.2. [10] *An $(2n+1)$-dimensional Sasakian manifold M is said to be M-projectively flat if the ∗-Weyl curvature tensor vanishes identically (that is, $\overline{W}^* = 0$).*

Definition 2.3. [10] *An $(2n+1)$-dimensional Sasakian manifold M is said to be ϕ-∗-Weyl flat if $g(\overline{W}^*(\phi X, \phi Y)\phi Z, \phi V) = 0$ for all X, Y, Z and $V \in \chi(M)$.*

Definition 2.4. [10] *An $(2n+1)$-dimensional Sasakian manifold M is said to be ξ-∗-Weyl flat if $\overline{W}^*(X, Y)\xi = 0$ for all $X, Y \in \chi(M)$.*

3. Some properties of Sasakian manifold admitting Zamkovoy connection

In this section we consider the $*$-Weyl curvature tensor of Sasakian manifold admitting Zamkovoy connection.
Using (7) and (9), in (2) yields,

$$\overline{\nabla}_X Y = \nabla_X Y + g(X,\phi Y)\xi + \eta(Y)\phi X + \eta(X)\phi Y. \tag{17}$$

We now calculate the Riemann curvature tensor $\overline{R}$ using (17) as follows:

$$\begin{aligned} \overline{R}(X,Y)Z &= R(X,Y)Z - g(Z,\phi X)\phi Y - g(Y,\phi Z)\phi X - 2g(\phi X,Y)\phi Z \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y. \end{aligned} \tag{18}$$

Using (10) and taking $Z=\xi$ in the above equation, we get

$$\overline{R}(X,Y)\xi = 0. \tag{19}$$

On contracting (18), we obtain the Ricci tensor $\overline{Ric}$ of a Sasakian manifold with respect to the Zamkovoy connection as

$$\overline{Ric}(Y,Z) = Ric(Y,Z) + 2g(Y,Z) - 2(n+1)\eta(Y)\eta(Z), \tag{20}$$

this gives

$$\overline{Q}Y = QY + 2Y - 2(n+1)\eta(Y)\xi. \tag{21}$$

Contracting with respect to Y and Z in (20), we get

$$\overline{r} = r + 2n, \tag{22}$$

where $\overline{r}$ and r are the scalar curvatures with respect to the Zamkovoy connection and the Levi-Civita connection respectively.

Theorem 3.1. *If a sasakian manifold M is Ricci flat with respect to the Zamkovoy connection then M is an η-Einstein manifold.*

Proof. Suppose that the Sasakian manifold is Ricci flat with respect to the Zamkovoy connection. Then from (20), we get

$$Ric(Y,Z) = -2g(Y,Z) + 2(n+1)\eta(Y)\eta(Z).$$

This shows that M is an η-Einstein manifold. □

Theorem 3.2. *A $*$-Weyl flat Sasakian manifold M admitting Zamkovoy connection $\overline{\nabla}$ is an η-Einstein manifold.*

Proof. We assume that the manifold M with respect to the Zamkovoy connection is $*$-Weyl flat that is $\overline{W^*}=0$. Then from (3), (14) and (15):

$$\begin{aligned} \overline{R}(X,Y)Z &= \frac{1}{(2n-1)}\{(4-2n)g(Y,Z)X + (2n-4)g(X,Z)Y + \eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(Y,Z)QX - g(X,Z)QY - (2n+3)g(Y,Z)\eta(X)\xi \\ &+ (2n+3)g(X,Z)\eta(Y)\xi\} - \frac{(r+2n-4n^2)}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned}$$

Taking the inner product of the above equation with V, we have

$$\begin{aligned} g(\overline{R}(X,Y)Z,V) &= \frac{1}{(2n-1)}\{(4-2n)g(Y,Z)g(X,V) \\ &+ (2n-4)g(X,Z)g(Y,V)+\eta(X)\eta(Z)g(Y,V)-\eta(Y)\eta(Z)g(X,V) \\ &+ g(Y,Z)g(QX,V)-g(X,Z)g(QY,V)-(2n+3)g(Y,Z)\eta(X)\eta(V) \\ &+ (2n+3)g(X,Z)\eta(Y)\eta(V)\}-\frac{(r+2n-4n^2)}{2n(2n-1)}\{g(Y,Z)g(X,V) \\ &- g(X,Z)g(Y,V)\}. \end{aligned} \tag{23}$$

On contracting (23) over X and V, we get

$$Ric(Y,Z)=\frac{(8n^2-2n-2r+1)}{2(n-1)}g(Y,Z)+\frac{(4n^2+2n+1)}{2(n-1)}\eta(Y)\eta(Z).$$

Therefore M is an η-Einstein manifold. □

Theorem 3.3. *A ξ-∗-Weyl flat Sasakian manifold M admitting Zamkovoy connection $\overline{\nabla}$ is an η-Einstein manifold.*

Proof. Now, we assume that the manifold M with respect to the Zamkovoy connection is ξ-∗-Weyl flat, that is, $\overline{W^*}(X,Y)\xi=0$. Then it follows (3), (14) and (15), it follows that

$$\frac{1}{(2n-1)}\{(1-2n)\eta(Y)X+(2n-1)\eta(X)Y-\eta(X)\overline{Q}Y+\eta(Y)\overline{Q}X\} - \left(\frac{r+2n-4n^2}{2n(2n-1)}\right)(\eta(Y)X-\eta(X)Y)=0,$$

now, taking inner product with a vector field V, we get

$$\frac{1}{(2n-1)}\{(2n-1)\eta(X)g(Y,V)-(2n-1)\eta(Y)g(X,V)-\eta(X)g(\overline{Q}Y,V) +\eta(Y)g(\overline{Q}X,V)\}-\left(\frac{r+2n-4n^2}{2n(2n-1)}\right)(\eta(Y)g(X,V)-\eta(X)g(Y,V)=0,$$

setting $Y=\xi$ and using (20) in above equation, we have

$$Ric(X,V)=\frac{(r-4n)}{2n}g(X,V)+\frac{4n^2+4n-r}{2n}\eta(X)\eta(V),$$

therefore, M is an η-Einstein manifold. □

Theorem 3.4. *An $(2n+1)$-dimensional Sasakian manifold M is ξ-∗-Weyl flat with respect to Zamkovoy connection iff it is so with respect Levi-Civita connection, provided that vector fields are horizantal vector fields.*

Proof. Using (1), (2), (18), (20),(21) and (22), yields

$$\begin{aligned}\overline{W}^*(X,Y)Z &= W^*(X,Y)Z - g(Z,\phi X)\phi Y - g(Y,\phi Z)\phi X \qquad (24)\\ &- 2g(\phi X,Y)\phi Z + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi - \eta(Y)\eta(Z)X\\ &+ \eta(X)\eta(Z)Y - \frac{1}{(2n-1)}\{2g(Y,Z)X - 2(n+1)\eta(Y)\eta(Z)X\\ &- 2g(X,Z)Y + 2(n+1)\eta(X)\eta(Z)Y + 2g(Y,Z)X\\ &- 2(n+1)g(Y,Z)\eta(X)\xi - 2g(X,Z)Y + 2(n+1)g(X,Z)\eta(Y)\xi\}\\ &+ \frac{2n}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}.\end{aligned}$$

Setting $Z=\xi$ in (24), we have

$$\begin{aligned}\overline{W}^*(X,Y)\xi &= W^*(X,Y)\xi - \eta(Y)X + \eta(X)Y\\ &- \frac{1}{(2n-1)}\{(1-2n)\eta(Y)X + (2n-1)\eta(X)Y\}.\end{aligned}$$

If X and Y are horizontal vector fields, then from above equation, it follows that $\overline{W}^*(X,Y)\xi = W^*(X,Y)\xi$.
Therefore, Sasakian manifold M is ξ-$*$-Weyl flat with respect to Zamkovoy connection iff it is so with respect Levi-Civita connection, provided that vector fields are horizantal vector fields. Hence, the theorem. □

Theorem 3.5. *A ϕ-$*$-Weyl flat Sasakian manifold M admitting Zamkovoy connection is an η-Einstein manifold.*

Proof. We assume that a Sasakian manifold M is ϕ-$*$-Weyl flat with respect to Zamkovoy connection, i.e, $g(\overline{W}^*(\phi X,\phi Y)\phi Z,\phi V) = 0$ for all $X,Y,Z,V \in \chi(M)$. Then, in view of (3), we have

$$\begin{aligned}0 &= g(\overline{R}(\phi X,\phi Y)\phi Z,\phi V) - \frac{1}{(2n-1)}\{\overline{Ric}(\phi Y,\phi Z)g(\phi X,\phi V)\\ &- \overline{Ric}(\phi X,\phi Z)g(\phi Y,\phi V) - 2(2n-1)g(\phi Y,\phi Z)g(\phi X,\phi V)\\ &+ 2(2n-1)g(\phi X,\phi Z)g(\phi Y,\phi V) + \overline{Ric}(\phi X,\phi V)g(\phi Y,\phi Z)\\ &- g(\phi X,\phi Z)\overline{Ric}(\phi Y,\phi V)\} + \frac{(r+2n-4n^2)}{2n(2n-1)}\{g(\phi Y,\phi Z)g(\phi X,\phi V)\\ &- g(\phi X,\phi Z)g(\phi Y,\phi V)\}.\end{aligned}$$

In a $(2n+1)$ dimensional almost contact metric manifold M if $\{e_1,e_2,...,e_n,\xi\}$ is a local orthonormal basis of vector fields in M then $\{\phi e_1,\phi e_2,...,\phi e_n,\xi\}$ is also a local orthonormal basis and putting $Y=Z=e_i$ in preciding equation and summing up with respect to i $1\le i\le 2n+1$, we obtain

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z),$$

where $a = \frac{\{-8n^3+12n^2-10n+r\}}{2n}$ and $b = \frac{\{8n^3-8n^2+10n-r\}}{2n}$.
Which is η-Einstein manifold. □

Proposition 3.1. In an $(2n+1)$-dimensional Sasakian manifold M admitting Zamkovoy connection ∇, if the condition $\overline{W}^*(\xi,U)\circ\overline{R} = 0$ holds, then the equation: $Ric^2(Y,U) = \frac{(4n^2-8n+r)}{2n}g(Y,U) + \frac{(4n^2-4n+r)}{2n}\eta(Y)\eta(U)$ is satisfied on M.

Proof. We consider a Sasakian manifold M satisfying the condition:

$$(\overline{W}^*(\xi,U)\circ\overline{R})(X,Z)V=0,$$

where $\overline{W}$ and $\overline{R}$ denote the M-projective curvature tensor and the Riemannian curvature tensor with respect to the Zamkovoy connection, respectively. For all $U,Y,Z\in\chi(M)$, we have

$$\begin{aligned}\overline{W}^*(\xi,U)\overline{R}(X,Z)V &= \overline{R}(\overline{W}^*(\xi,U)X,Z)V+\overline{R}(X,\overline{W}^*(\xi,U)Z)V \\ &+ \overline{R}(X,Z)\overline{W}^*(\xi,U)V.\end{aligned} \tag{25}$$

Replacing X by ξ in (25) and using (19), we get

$$\overline{R}(\overline{W}^*(\xi,U)\xi,Z)V=0,$$

with the help of (3), (19), (14), (15) and preceding equation, we get

$$\frac{(-4n^2+4n-r)}{2n(2n-1)}\overline{R}(U,Z)V-\frac{1}{(2n-1)}\overline{R}(QU,Z)V=0.$$

Taking inner product with Y in the above equation and considering a frame field of M and contracting over Z and V, we get

$$\begin{aligned}Ric^2(Y,U) &= \frac{(4n^2-8n+r)}{2n}Ric(Y,U)+\frac{(4n^2-4n+r)}{2n}g(Y,U) \\ &+ \frac{(8n^3+12n^2-4n+r)}{2n}\eta(Y)\eta(U).\end{aligned}$$

Hence, the theorem is proved. □

Example 1. We consider the 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3\}$, where (x,y,z) are standard coordinates in $\mathbb{R}^3$. Let e_1, e_2, e_3 be a linearly independent frame field on M, given by,

$$e_1=e^z\frac{\partial}{\partial x},\quad e_2=e^z\frac{\partial}{\partial y},\quad e_3=\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by,

$$g(e_i,e_j)=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j.\end{cases}$$

Let η be the 1-form by $\eta(X)=g(X,e_3)$ for any $X\in\chi(M^3)$, and ϕ be the $(1,1)$-tensor field defined by,

$$\phi e_1=e_1,\quad \phi e_2=e_2,\quad \phi e_3=0.$$

By direct computations, we can easily to see that

$$\phi^2X=-X+\eta(X)\xi,\quad \eta(X)=g(X,\xi),$$

and

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y).$$

for all $X,Y\in\chi(M^3)$. Thus $M^3(\phi,\xi,\eta,g)$ is a 3-dimensional Sasakian manifold. From the Lie-operatory, we have the non-zero compnents

$$[e_1,e_2]=0,\quad [e_1,e_3]=-e_1,\quad [e_2,e_3]=-e_2.$$

Furthermore, by ∇, we denote the Levi-civita connection on M, by using Koszul's formula, we can calculate, easily

$$\begin{aligned} \nabla_{e_1}e_1 &= e_3, \quad & \nabla_{e_2}e_1 &= 0, \quad & \nabla_{e_3}e_1 &= 0,\\ \nabla_{e_1}e_2 &= 0, \quad & \nabla_{e_2}e_2 &= e_3, \quad & \nabla_{e_3}e_2 &= 0,\\ \nabla_{e_1}e_3 &= -e_1, \quad & \nabla_{e_2}e_3 &= -e_2, \quad & \nabla_{e_3}e_3 &= 0. \end{aligned}$$

From the above results we see that the structure (ϕ, ξ, η, g) satisfies

$$\nabla_X \xi = -\phi^2 X,$$

hence (ϕ, ξ, η, g) is a Sasakian structure and $M^3(\phi, \xi, \eta, g)$ is a 3-dimensional Sasaskian manifold. We obtain the components of the curvature tensor as follows:

$$\begin{aligned} R(e_1,e_2)e_1 &= e_2, \quad & R(e_1,e_3)e_1 &= e_3, \quad & R(e_2,e_3)e_1 &= 0,\\ R(e_1,e_2)e_2 &= -e_1, \quad & R(e_1,e_3)e_2 &= 0, \quad & R(e_2,e_3)e_2 &= e_3,\\ R(e_1,e_2)e_3 &= 0, \quad & R(e_1,e_3)e_3 &= -e_1, \quad & R(e_2,e_3)e_3 &= -e_2. \end{aligned}$$

From the above relations, we have

$$Ric(e_1,e_1) = Ric(e_2,e_2) = Ric(e_3,e_3) = -2.$$

Using (17), we obtain

$$\begin{aligned} \overline{\nabla}_{e_1}e_1 &= 2e_3, \quad & \overline{\nabla}_{e_2}e_1 &= 0, \quad & \overline{\nabla}_{e_3}e_1 &= e_1,\\ \overline{\nabla}_{e_1}e_2 &= 0, \quad & \overline{\nabla}_{e_2}e_2 &= 2e_3, \quad & \overline{\nabla}_{e_3}e_2 &= e_2,\\ \overline{\nabla}_{e_1}e_3 &= 0, \quad & \overline{\nabla}_{e_2}e_3 &= 0, \quad & \overline{\nabla}_{e_3}e_3 &= 0. \end{aligned}$$

The non zero components of Riemannian curvature tensor with respect to Zamkovoy connection are given by:

$$\begin{aligned} \overline{R}(e_1,e_2)e_1 &= 0, \quad & \overline{R}(e_1,e_3)e_1 &= 4e_3, \quad & \overline{R}(e_2,e_3)e_1 &= 0,\\ \overline{R}(e_1,e_2)e_2 &= 0, \quad & \overline{R}(e_1,e_3)e_2 &= 0, \quad & \overline{R}(e_2,e_3)e_2 &= 4e_3,\\ \overline{R}(e_1,e_2)e_3 &= 0, \quad & \overline{R}(e_1,e_3)e_3 &= 0, \quad & \overline{R}(e_2,e_3)e_3 &= 0. \end{aligned}$$

Using the above curvature tensors with respect to ∇ and $\overline{\nabla}$, the relation (19) can be verified.

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