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OPTIMIZATION METHODS FOR THE QUADRATIC EIGENVALUE ASSIGNMENT PROBLEM WITH BOUNDED FEEDBACK CONTROLLER

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Abstract. This work considers the quadratic eigenvalue assignment problem for vibrating structures by state feedback. A nonlinear least–squares formulation of the problem with a semismooth objective function is considered. A Levenberg–Marquardt method that uses a nonmonotone trust region combined with a line search backtracking strategy is proposed to tackle the problem. Global convergence and local superlinear/quadratic convergence rates of the algorithm are established. Moreover, a logarithmic barrier interior–point method is addressed to tackle an inequality constrained problem resulting from incorporating an upper bound on the computed feedback controllers. Numerical results are given to demonstrate the performance of proposed methods.

1. Introduction

In this article, we consider the linear time–invariant quadratic control system

$$M\ddot{x}(t) + D\dot{x}(t) + Nx(t) = Bu(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) = \mathbb{R}^p$ is the control vector, and $M, D, N \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ are given constant matrices, where M is assumed to be nonsingular. Models of the form (1) frequently arise in a wide range of applications in vibration and structural analysis. In particular, the quadratic eigenvalue assignment problem (QEAP) is the focus of this work, see e.g., the survey [24] as well as [1, 2, 3, 4, 5, 8, 10, 14, 15, 17, 25, 26, 27]. Several studies have proposed various optimization–based methods for solving the partial QEAP [1, 2, 4, 5, 13, 14, 26]. The algorithms considered in the articles [1, 2, 4, 14] are mainly for computing feedback controllers of the QEAP with minimum norm. Different approaches based on the method of receptance have been considered by [2, 3, 25, 26, 30]. Algorithms based on multi–step methods have been introduced in the articles [15, 16]. An

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LMI approach was considered in the work by [12]. Moreover, a number of studies have employed different algorithms to tackle the QEAP in [8, 10, 17].

The following control vector is often used to close the system (1)

$$u(t) = K_1\dot{x}(t) + K_2x(t), \tag{2}$$

which implies the closed system

$$M\ddot{x}(t) + (D - BK_1)\dot{x}(t) + (N - BK_2)x(t) = 0, \tag{3}$$

where K_1 and $K_2 \in \mathbb{R}^{p\times n}$ are the feedback matrices.

The second–order system (3) is transformed into first order as given by the following lemma.

Lemma 1.1. Assuming that M is nonsingular, then the quadratic control system (3) can be reduced to the following closed–loop system

$$\dot{z}(t) = Az(t) + \hat{B}u(t) = (A + \hat{B}K)z(t) \tag{4}$$

where

$$A = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -M^{-1}N & -M^{-1}D \end{bmatrix},\ \hat{B} = \begin{bmatrix} 0_{n\times p} \\ M^{-1}B \end{bmatrix},\ K = [K_2\ K_1],\ z(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}, \tag{5}$$

or equivalently as

$$\dot{z}(t) = A_c(K_1, K_2)z(t), \tag{6}$$

where

$$A_c(K_1, K_2) = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -M^{-1}(N - BK_2) & -M^{-1}(D - BK_1) \end{bmatrix}.$$

To address the proposed optimization problem, let $\hat{\lambda}_1, \ldots, \hat{\lambda}_{2n} \in \mathbb{C}$ be given desired eigenvalues of the closed–loop system (3) which are closed under conjugation. The QEAP is to find feedback matrices K_1 and $K_2 \in \mathbb{R}^{p\times n}$ that fulfill the system

$$\lambda_i(A_c(K_1, K_2)) = \hat{\lambda}_i, \qquad i = 1, \ldots, 2n, \tag{7}$$

where $\lambda_i(A_c(K_1, K_2))$ are the eigenvalues of the closed–loop system matrix $A_c(K_1, K_2)$.

Let $r : \mathbb{R}^{2(p\times n)} \to \mathbb{C}^{2n}$ be the residual vector defined as

$$r(K_1, K_2) = \begin{pmatrix} \lambda_1(A_c(K_1, K_2)) - \hat{\lambda}_1 \\ \vdots \\ \lambda_{2n}(A_c(K_1, K_2)) - \hat{\lambda}_{2n} \end{pmatrix}. \tag{8}$$

The QEAP is to find matrices K_1 and K_2 that solve the nonlinear least-squares problem

$$\min_{K_1, K_2\in\mathbb{R}^{p\times n}} f(K_1, K_2) = \frac{1}{2}\big|\big|r(K_1, K_2)\big|\big|^2 = \frac{1}{2}\sum_{i=1}^{2n} r_i(K_1, K_2)^*\, r_i(K_1, K_2), \tag{9}$$

where the superscript $*$ denotes the conjugate transpose of a complex number. To simplify the presentation let $K = [K_2\ K_1] \in \mathbb{R}^{p\times 2n}$ and let $\kappa \in \mathbb{R}^m$ be the vector obtained by stretching K into the vector $\kappa \in \mathbb{R}^m$, where $m = 2n\,.\,p$. Consequently, Problem (9) is written as

$$\min_{\kappa\in\mathbb{R}^m} f(\kappa) = \frac{1}{2}\|r(\kappa)\|_2^2 = \frac{1}{2}\sum_{i=1}^{2n} r_i(\kappa)^*\, r_i(\kappa), \tag{10}$$

where

$$r(\kappa) = (r_1(\kappa), \ldots, r_{2n}(\kappa))^T, \tag{11}$$

is the residual vector given by (8).

One of the important issues when solving this problem is to replace some undesirable eigenvalues by assigning them to their corresponding desired values and to leave the rest unaltered. The problem in this case is known as the partial quadratic eigenvalue assignment; see, e.g., [3, 5, 10, 14, 15, 26]. The considered least-squares approach, however, has

the freedom to select part or all eigenvalues of the system to be assigned based on the chosen desired eigenvalues $\hat{\lambda}_i$'s. So, the considered approach clearly belongs to the partial eigenvalue assignment category.

Note that available general–purpose methods like the Matlab solvers `lsqnonlin` or `fminunc` cannot solve Problem (10), where derivatives of the objective function that includes eigenvalues are required.

The objective function $f(\kappa)$ holds eigenvalues, however it is known that eigenvalues are not differentiable at points where repeated eigenvalues exist, see e.g. [18]. Since an eigenvalue of a symmetric matrix can be written as the difference of two convex functions, then this implies that eigenvalues are semismooth functions; see [23]. This also allows to assume that the objective function $f(\kappa)$ is semismooth; see among others [20, 21, 23, 29] for related works on semismooth problems.

This research primarily aims to apply one of the ideal choices for solving the least-squares problem (10) and to determine how large the quadratic control system (1) can be tackled as it is converted into first order. Therefore, Levenberg–Marquardt method (LM) comes as the first choice, because it encompasses various characteristics mainly it uses only the convenient and often effective approximation of the objective function Hessian and it handles the rank deficiency of this approximate Hessian efficiently, see e.g. [6, 7, 21, 29]. In this research an LM method combined with trust region and line search globalization is proposed to tackle Problem (10). Since the standard LM method uses trust region, global convergence and local superlinear/quadratic convergence rates are established for the proposed algorithm under standard assumptions.

To enhance the quality of achieved feedback controllers for example by computing feedback gain matrices whose norms are bounded, the following inequality constrained problem is considered:

$$\text{(CP)} \quad \begin{cases} \min & f(\kappa) = \frac{1}{2}\sum_{i=1}^{2n} r_i(\kappa)^* \, r_i(\kappa), \\ \text{s.t.} & h(\kappa) = u_b - \|\kappa\|_2^2 \geq 0, \end{cases} \tag{12}$$

where $u_b > 0$ is a constant. A well–known approach to tackle this problem is to convert it into the following logarithmic barrier subproblem:

$$\text{(BP)} \quad \min \ \phi^{\hat{\mu}}(\kappa) = f(\kappa) - \hat{\mu}\log(h(\kappa)), \tag{13}$$

where $\hat{\mu} > 0$ is the barrier parameter. A logarithmic barrier interior–point method solves the constrained problem (12) by solving approximately the barrier subproblem (13) for a decreasing sequence converging to zero of the barrier parameters; see e.g., [19]. This method exhibits asymptotic quadratic rate of convergence. Moreover, it allows for various solution methods for the underlying linear algebraic equations obtained from the linearized optimality conditions at each iteration. For large–scale problems a quasi–Newton updated is considered to approximate the Hessian of $\phi^{\hat{\mu}}(\kappa)$.

The subsequent sections of this research are organized as follows. In the next subsection some preliminary and basic results are introduced for semismooth functions. In Subsection 1.2 first and second derivatives of the objective function are obtained. In Section 2 an LM method is described for finding a local solution of the least–squares problem (10). In Sections 3 and 4 the global convergence and local superlinear/quadratic rate of convergence of LM method are establish, respectively. In Section 5 a logarithmic barrier method is proposed to tackle the constrained problem (12). Section 6 is devoted to the numerical results, where the considered methods are tested on various test problems collected from different sources. The article ends with an appendix including the required derivatives of the objective function followed by some concluding remarks.

Notations: The symbol $\|\cdot\|$ denotes the Euclidean norm or the subordinate matrix norm. The eigenvalues of a matrix $M \in \mathbb{R}^{n\times n}$ are denoted by $\lambda_i(M),\ i = 1, \ldots, n$. To simplify the presentation and as mentioned above the matrix $K = [K_2 \ K_1] \in \mathbb{R}^{2(p\times n)}$ denotes the augmented feedback gain matrix. Moreover, $\kappa \in \mathbb{R}^m$ is the vector obtained by

stretching K into a column vector, where $m = 2np$. Sometimes the argument is skipped from writing like g_k is written instead of $g(\kappa_k)$.

1.1. **Preliminaries.** The following are some related definitions and basic results for semismooth functions.

Definition 1.1. (see [29]) Let $h : \mathbb{R}^m \to \mathbb{R}^n$ be locally Lipschitz continuous function and let $\mathcal{D}_h$ be the set of all κ such that h is differentiable. The generalized Jacobian of h at the point κ in the sense of Clark is defined by $\partial h(\kappa) = \mathrm{co}(\partial_B h(\kappa))$, where 'co' means convex hull and

$$\partial_B\, h(\kappa) = \left\{ M \in \mathbb{R}^{n\times m} \ : \ M = \lim_{\kappa_k \to \kappa} \nabla h(\kappa_k)^T,\ \kappa_k \in D_h \right\},$$

is called the B–differential of h at κ.

Definition 1.2. (see [23]) A locally Lipschitz function $h : \mathbb{R}^m \to \mathbb{R}^n$ is semismooth (respectively, strongly semismooth) at $\kappa \in \mathbb{R}^m$ if it is directionally differentiable at κ and for any $M \in \partial h(\kappa + d)$ it holds that

$$h(\kappa + d) - h(\kappa) - Md = o(\|d\|)\ (O(\|d\|^2)),$$

where the notation $G = o(\alpha)$ means $\|G\|/|\alpha| \to 0$ as $\alpha \to 0$.

The following helping result is needed later on (see, [29, Proposition 2.1]).

Lemma 1.2. *Let $\{\kappa_k\} \subset \mathbb{R}^m$ be a convergent sequence with a limit point $\kappa_* \in \mathbb{R}^m$. Then the following statements are satisfied:*

(*a*) *The residual function $r(\kappa)$ is semismooth, which implies that for any $J_k \in \partial_C\, r(\kappa_k)$,*

$$\|r(\kappa_k) - r(\kappa_*) - J_k(\kappa_k - \kappa_*)\| = o(\|\kappa_k - \kappa_*\|).$$

(*b*) *If $J(\kappa)$ is Lipschitz continuous, then the function $r(\kappa)$ is strongly semismooth. This implies that for any $J_k \in \partial_C\, r(\kappa_k)$ we have*

$$\|r(\kappa_k) - r(\kappa_*) - J_k(\kappa_k - \kappa_*)\| = O(\|\kappa_k - \kappa_*\|^2).$$

Let $\mathcal{D} \subseteq \mathbb{R}^m$ be the set of all $\kappa \in \mathbb{R}^m$ where f is differentiable. Since the set $\mathcal{D}$ is open, then it is convenient to replace it by the following subset:

$$\Omega(\kappa_0) := \{\kappa \in \mathcal{D} \ : \ f(\kappa) \le f(\kappa_0)\}, \tag{14}$$

where $\kappa_0 \in \mathbb{R}^m$ is given, and it is assumed that $\Omega(\kappa_0)$ is compact.

1.2. **Required derivatives.** Assume that the function $\Lambda(A_c(\kappa)) := \mathrm{diag}(\lambda_1, \ldots, \lambda_{2n})$ is differentiable and let $Q(A_c(\kappa)) \in \mathbb{C}^{2n\times 2n}$ be an orthogonal matrix whose columns q_i's are the eigenvectors of A_c that satisfy

$$q_i^T q_i = 1,$$
$$q_i^T A_c(\kappa) q_i = \lambda_i(\kappa), \quad i = 1, \ldots, 2n.$$

Then it holds that

$$\frac{\partial \lambda_i(\kappa)}{\partial \kappa_k} = q_i^T \frac{\partial A_c(\kappa)}{\partial \kappa_k} q_i, \quad i = 1, \ldots, 2n,\ k = 1, \ldots, m. \tag{15}$$

First and second–order derivatives of the objective function (10) are given by the following two lemmas; see [18, Lemmas 3.2 and 3.3] for the proof of similar results.

Lemma 1.3. *Let $\Lambda(A_c(\kappa))$ and Q be as defined above. Then the gradient vector of the objective function f is given by*

$$g(\kappa) = \mathrm{Re}\left\{J(\kappa)^T r(\kappa)\right\}, \tag{16}$$

where $\mathrm{Re}\{\cdot\}$ *denotes the real–part of a complex number,* $r(\kappa) = (\lambda - \hat{\lambda}) \in \mathbb{C}^{2n}$ *is the vector of residuals given by* (8)*, and* $J(\kappa)$ *is the* $2n \times m$ *Jacobian matrix:*

$$J(\kappa) = \begin{bmatrix} (q_1^T \hat{B} e_1 e_1^T q_1) & \dots & (q_1^T \hat{B} e_p e_{2n}^T q_1) \\ \vdots & \ddots & \vdots \\ (q_{2n}^T \hat{B} e_1 e_1^T q_{2n}) & \dots & (q_{2n}^T \hat{B} e_p e_{2n}^T q_{2n}) \end{bmatrix}, \tag{17}$$

where the $j := (k,l)$*th component of* $g(\kappa)$ *is given by:*

$$g_j(\kappa) = \mathrm{Re}\left\{ q_1^T \hat{B} e_k e_l^T q_1 (\lambda_1(A_c) - \hat{\lambda}_1) + \cdots + q_{2n}^T \hat{B} e_k e_l^T q_{2n} (\lambda_{2n}(A_c) - \hat{\lambda}_{2n}) \right\},\ j = 1, \dots, m \tag{18}$$

and e_i *is a vector with 1 at the* i*th position and all other entries are zeros.*

Lemma 1.4. *Let* $\Lambda(A_c(\kappa))$ *and* Q *be as defined above. Then the Hessian of the objective function* f *is given by*

$$\nabla^2 f(\kappa) = \mathrm{Re}\left\{ J(\kappa)^* J(\kappa) + \sum_{i=1}^{2n} r_i(\kappa)^* \Big(\frac{\partial^2 r_i(\kappa)}{\partial \kappa_k \partial \kappa_l} \Big) \right\}, \tag{19}$$

where $J(\kappa)$ *is given by* (17) *and* $\partial^2 r_i(\kappa)/\partial\kappa_k \partial\kappa_l$ *are the second derivative terms given by*

$$\frac{\partial^2 r_i(\kappa)}{\partial \kappa_k \partial \kappa_l} = \sum_{j=1, j\neq i}^{2n} \frac{[(q_i^T \hat{B} e_k e_l^T q_j)(q_j^T \hat{B} e_k e_l^T q_i) + (q_j^T \hat{B} e_k e_l^T q_i)(q_i^T \hat{B} e_k e_l^T q_j)]}{\lambda_i(A_c(\kappa)) - \lambda_j(A_c(\kappa))},\ i = 1, \dots, 2n. \tag{20}$$

Note that the required term of the Hessian of f for Gauss–Newton or LM methods is the following

$$H(\kappa) = \mathrm{Re}\left\{ J(\kappa)^* J(\kappa) \right\}, \tag{21}$$

where the second term is not required.

2. Levenberg–Marquardt method for QEAP

Gauss–Newton and Levenberg–Marquardt methods come as the first choice when solving the nonlinear least–squares problem (10). Gauss–Newton method for instance as it solves Problem (10) requires the solution of the following algebraic system of linear equations every iteration:

$$H(\kappa_k) d = -\nabla f(\kappa_k), \tag{22}$$

where $\nabla f(\kappa_k) = \mathrm{Re}\big(J(\kappa_k)^* r(\kappa_k)\big)$ and $H(\kappa_k) = \mathrm{Re}\big(J(\kappa_k)^* J(\kappa_k)\big)$ are given by (16) and (21), respectively, $r(\kappa_k)$ is the residual vector (11) and $J(\kappa_k)$ is the Jacobian matrix (17). However, if $J(\kappa_k)$ is rank–deficient, Gauss–Newton method might fail. To avoid such a drawback one possible way is to consider instead a Levenberg–Marquardt method. The quadratic model for this method is defined as

$$\widetilde{m}_k(d) = \frac{1}{2}\|r_k\|^2 + r_k^T J(\kappa_k)^T d + \frac{1}{2} d^T H(\kappa_k) d. \tag{23}$$

Levenberg–Marquardt method is based on solving for $d \in \mathbb{R}^m$ the following algebraic system of equations every iteration

$$(H(\kappa_k) + \mu_k I) d = -\nabla f(\kappa_k), \tag{24}$$

where $\mu_k > 0$ is the LM parameter, $H(\kappa_k)$ and $\nabla f(\kappa_k)$ are as defined above. Various updates for μ_k are considered in the literature; see e.g., [7] among them are the two updates $\mu_k = \beta_k \|r(\kappa_k)\|$ and $\mu_k = \beta_k \|r(\kappa_k)\|^2$, where $\beta_k > 0$ is given. In the current work the first one is considered for updating μ_k.

The following quantities of the actual and predicted reductions of the objective function their role is to accept the computed step d of the system (24):

$$ared_k(d) = \Gamma_k - f(\kappa_k + d), \tag{25}$$

$$pred_k(d) = -\nabla f(\kappa_k)^T d - \frac{1}{2} d^T H(\kappa_k) d, \tag{26}$$

where

$$\Gamma_k = \begin{cases} f_k & \text{if } k = 0 \\ (\zeta_{k-1} E_{k-1} \Gamma_{k-1} + f_k)/E_k & \text{if } k \geq 1 \end{cases} \tag{27}$$

and

$$E_k = \begin{cases} 1 & \text{if } k = 0 \\ \zeta_{k-1} E_{k-1} + 1 & \text{if } k \geq 1. \end{cases} \tag{28}$$

Define the ratio

$$\rho_k = \frac{ared_k(d)}{pred_k(d)}, \tag{29}$$

where according to the value of ρ_k the computed step d is accepted, otherwise a backtracking line–search rule is employed according to the following sufficient decrease condition: Compute i_k the least non–negative integer i satisfying

$$f(\kappa_k + \sigma^i d_k) \leq \Gamma_k + \delta \sigma^i \nabla f(\kappa_k)^T d_k, \tag{30}$$

and set $\kappa_{k+1} = \kappa_k + \alpha_k d_k$ with $\alpha_k = \sigma^{i_k}$, where $\sigma \in (0,1)$ and $\delta \in (0,1)$ are given constants.

The algorithm of LM for solving the QEAP is stated in the following lines.

Algorithm 2.1. (Nonmonotone LM method for the QEAP)
Let M, D, N, and B be given constant matrices of the control system (1) and $\hat{\lambda}_i$, $i = 1, \ldots, 2n$ be the desired eigenvalues. Form the matrices A and $\hat{B}$ according to (5).
Choose $0 < \eta_1 < \eta_2 < 1$, $0 < \gamma_1 < 1 < \gamma_2$, $0 \leq \zeta_{\min} < 1$, $\zeta_{\min} \leq \zeta_{\max} < 1$, $0 < \delta < 1$, $0 < \sigma < 1$. Let $\beta_0 = \mu_0 > \bar{\mu} > 0$, $\zeta_{\min} \leq \zeta_0 \leq \zeta_{\max}$, $0 < \epsilon^{\text{tol}} < 1$ be given. Set $K_0 = [K_2^0 \; K_1^0]$ for the given starting feedback matrices K_1^0 and K_2^0 and stretch K_0 into the column vector κ_0. Compute $r(\kappa_0)$, $f(\kappa_0)$, and $J(\kappa_0)$.
For $k = 0, 1, 2, \ldots$

1. Solve for d_k the algebraic system of equations (24).
2. Compute the quantities $ared_k(d)$ and $pred_k(d)$ by (25)–(28), and set
$$\rho_k = \frac{ared_k(d_k)}{pred_k(d_k)}.$$
3. If $\rho_k \geq \eta_1$, set $\kappa_{k+1} = \kappa_k + d_k$ and go to Step 4. Otherwise, compute the step–length $\alpha_k > 0$ that satisfies the sufficient decrease condition (30) and set $\kappa_{k+1} = \kappa_k + \alpha_k d_k$.
4. Choose $\zeta_k \in [\zeta_{\min}, \zeta_{\max}]$ and update β_{k+1} by
$$\beta_{k+1} = \begin{cases} \gamma_2 \beta_k & \text{if } \rho_k \leq \eta_1 \\ \beta_k & \text{if } \rho_k \in (\eta_1, \eta_2) \\ \max\{\gamma_1 \beta_k, \bar{\mu}\} & \text{if } \rho_k \geq \eta_2. \end{cases}$$
5. Compute $f(\kappa_{k+1})$ by (8) & (10), $J(\kappa_{k+1})$ by (17), $\nabla f(\kappa_{k+1})$ by (16), and set $\mu_{k+1} = \beta_{k+1} \|r(\kappa_{k+1})\|$.
6. If $\|\nabla f(\kappa_{k+1})\| \leq \epsilon^{\text{tol}}$, stop; otherwise compute $H(\kappa_{k+1})$ by (21), set $k \leftarrow k+1$ go to Step 1.

End (For)

Remark 1. *The main objective when solving Problem* (10) *is to achieve some* κ_k *such that* $f(\kappa_k)$ *converges to zero. Therefore, a suitable alternative stopping condition might be*

$$\sqrt{f(\kappa_k)} < \epsilon^{\text{tol}}, \tag{31}$$

where $\epsilon^{\text{tol}} \in (0,1)$.

The matrix $H(\kappa_k) + \mu_k I$ is symmetric and positive definite. So, the system (24) has a solution and Step 1 of Algorithm 2.1 is well–defined. In order to prove that Algorithm 2.1 is well–defined it is assumed that the algorithm does not terminate after a finite number of iterations, which means that

$$g(\kappa_k) \neq 0 \quad \forall\ k \geq 0. \tag{32}$$

Moreover, the two index sets are defined by:

$$\mathcal{I} := \{\, k \,:\, \rho_k \geq \eta_1 \}, \quad \mathcal{J} := \{\, k \,:\, \rho_k < \eta_1 \}.$$

The following well–known lemma gives a lower bound of the predicted decrease.

Lemma 2.5. *Let d_k be computed by* (24)*. Then the following inequality holds for all $k \geq 0$*

$$pred_k = \widetilde{m}_k(0) - \widetilde{m}_k(d_k) \geq \frac{1}{2}\|g_k\| \ \min\left\{\|d_k\|, \frac{\|g_k\|}{\|H_k\|}\right\}. \tag{33}$$

Lemma 2.6. *Let $\{\kappa_k\}$ be generated by Algorithm 2.1. Then $f(\kappa_k) \leq \Gamma_k$ for all $k \geq 0$.*

Proof. If $k = 0$, the result follows by (27). For $k \geq 1$ let us consider the two cases: $k \in \mathcal{I}$ and $k \in \mathcal{J}$. First, the case when $k \in \mathcal{I}$ yields $\rho_k \geq \eta_1$, which by Lemma 2.5 and (24) implies that

$$\Gamma_k - f(\kappa_{k+1}) \geq \frac{1}{2}\eta_1 \, \|g_k\| \min\left\{\|d_k\|, \frac{\|g_k\|}{\|H_k\|}\right\} \geq 0.$$

So, $\Gamma_k \geq f(\kappa_{k+1})$. If $k \in \mathcal{J}$, then the computed trial step d will not be accepted and a backtracking line–search rule is applied. By (24) one has

$$g_k^T d_k = -d_k^T (H_k + \mu_k I) d_k \leq 0,$$

and by the definition of Γ_k and E_k one has

$$\begin{aligned} \Gamma_{k+1} &= (\zeta_k E_k \Gamma_k + f(\kappa_{k+1}))/E_{k+1} \\ &\geq (\zeta_k E_k f(\kappa_{k+1}) + f(\kappa_{k+1}))/E_{k+1} = f(\kappa_{k+1}) \end{aligned}$$

holds for all $k \geq 0$, which completes the proof. □

Lemma 2.7. *Let $\{\kappa_k\}$ be an infinite sequence generated by Algorithm 2.1. Then the algorithm is well-defined and the non–monotone line search rule* (58) *terminates after a finite number of iterations.*

Proof. By Lemma 2.5 one has $pred_k \geq 0$. To prove the first statement one needs to show that

$$pred_k \neq 0 \quad \forall\ k \geq 0.$$

By contradiction, let us assume that there exists an integer $i \geq 0$ such that $pred_i = 0$. Then from (32) and Lemma 2.5 one has $d_i = 0$. So, from (24) it follows that $\nabla f(\kappa_i) = 0$, which contradicts with (32).

Next, by Lemma 2.6 then $\Gamma_k \geq f(\kappa_k)$ holds for all $k \geq 0$. Consequently, from (24) it follows for all $d_k \neq 0$ that

$$\nabla f(\kappa_k)^T d_k = -d_k^T (H_k + \mu_k I) d_k < 0,$$

which means that d_k is descent direction for f at κ_k. This implies that Armijo backtracking rule terminates after a finite number of iterations, i.e., there exists an integer $i > 0$ such that

$$f(\kappa_k + \sigma_i d_k) \leq f(\kappa_k) + \delta \sigma^i \nabla f(\kappa_k)^T d_k.$$

□

3. Global convergence

The following assumptions are imposed to show the global convergence of Algorithm 2.1.

Assumption 3.1. *Assumed that:*

(*i*) *The level set $\Omega(\kappa_0)$ is closed and bounded.*
(*ii*) *The function $g(\kappa) = \nabla f(\kappa)$ is semismooth in $\Omega(\kappa_0)$.*
(*iii*) *The function $r(\kappa)$ is Lipschitz continuous in $\Omega(\kappa_0)$, i.e., there exists a constant $L > 0$ such that*

$$\|r(\xi) - r(\eta))\| \leq L\|\xi - \eta\| \quad \forall\ \xi,\ \eta \in \Omega(\kappa_0). \tag{34}$$

(*iv*) *There exists a constant $\nu_1 > 0$ such that*

$$d_k^T J_k^T J_k d_k \geq \nu_1 \|d_k\|^2 \quad \forall \quad d_k \in \mathbb{R}^n,\ k \geq 0. \tag{35}$$

It follows from (34) that

$$\|J(\kappa)\| \leq L \quad \forall\ \kappa \in \Omega(\kappa_0). \tag{36}$$

Lemma 3.8. *Let $\{\kappa_k\}$ be generated by Algorithm 2.1. Then $\{\kappa_k\}$ remain in $\Omega(\kappa_0)$. Moreover, the sequence $\{\Gamma_k\}$ is non-increasing monotonically and is convergent.*

Proof. If $k = 0$, the result obviously holds. Assume that $\kappa_k \in \Omega(\kappa_0)$ for all $k \geq 0$. Then $f(\kappa) \leq f(\kappa_0)$ for all $k \geq 0$. By the definition of Γ_k in (27) and (28) Γ_k is a convex combination of $f(\kappa_0)$, $f(\kappa_1), \ldots,$ $f(\kappa_k)$. So, $\Gamma_k \leq f(\kappa_0)$. From the proof of Lemma 2.6 we have $\Gamma_{k+1} \leq f(\kappa_0)$. Hence, $f(\kappa_{k+1}) \leq \Gamma_k \leq f(\kappa_0)$ which means that $\kappa_{k+1} \in \Omega(\kappa_0)$.

On the other hand, by the definition of Γ_k and $\Gamma_k \geq f(\kappa_{k+1})$ we have

$$\Gamma_{k+1} = (\zeta_k E_k \Gamma_k + f(\kappa_{k+1}))/E_{k+1} \leq (\zeta_k E_k \Gamma_k + \Gamma_k)/E_{k+1} = \Gamma_k.$$

So, $\{\Gamma_k\}$ is non-increasing monotonically and since $\Gamma_k \geq 0$, then $\{\Gamma_k\}$ is convergent. □

The following lemma shows that under the imposed assumptions the step-size α_k is bounded below.

Lemma 3.9. *Let Assumption 3.1 holds and the sequence $\{\kappa_k\}$ be generated by Algorithm 2.1. Then there exists a constant $0 < c < 1$ such that*

$$\alpha_k > c \quad \forall\ k \geq 0.$$

Proof. If $\alpha_k = 1$, then there is nothing to prove. Let $\alpha_k \leq 1$. From Step 3 of Algorithm 2.1 we have

$$f(\kappa_k + \alpha_k d_k) > \Gamma_k + \delta \alpha_k g_k^T d_k. \tag{37}$$

Since $g(\kappa)$ is semismooth in $\Omega(\kappa_0)$, then

$$f(\kappa_k + \alpha_k d_k) = f(\kappa_k) + \alpha_k g_k^T d_k + \frac{1}{2}\alpha_k^2 d_k^T G(\xi_k) d_k \tag{38}$$

where $G(\xi_k)$ is the generalized Jacobian of $g(\kappa)$ at κ_k and $\xi_k \in (\kappa_k, \kappa_k + \alpha_k d_k)$. Since $f(\kappa_k) \leq \Gamma_k$, then (37) and (38) imply that

$$-(1 - \delta) g_k^T d_k < \frac{1}{2}\alpha_k d_k^T G(\xi_k) d_k.$$

By Assumption 3.1 and Lemma 3.8 there exists a constant $\nu_2 > \nu_1$ such that

$$\|G(\xi)\| \leq \nu_2 \quad \forall\ \xi \in \Omega(\kappa_0),$$

which leads to

$$-(1 - \delta) g_k^T d_k < \frac{1}{2}\alpha_k \nu_2 \|d_k\|^2.$$

Since $pred_k \geq 0$, then (35) implies that

$$\frac{(1 - \delta)}{2}\nu_1 \|d_k\|^2 \leq \frac{1}{2}\alpha_k \nu_2 \|d_k\|^2. \tag{39}$$

The result follows from (39) by choosing $c = ((1 - \delta)\nu_1)/\nu_2$. □

Lemma 3.10. *Let Assumption 3.1 be satisfied and $\{\kappa_k\}$ be an infinite sequence generated by Algorithm 2.1. Then*

$$(1-\zeta_{\max})\gamma\,\|g_k\|\,\min\Big\{\|d_k\|,\frac{\|g_k\|}{\|H_k\|}\Big\}\le\Gamma_k-\Gamma_{k+1},\tag{40}$$

where $\gamma=\min\{\frac{1}{2}\eta_1,\frac{1}{2}\delta c\}$.

Proof. First, let $k\in\mathcal{I}$, which means that $\rho_k\ge\eta_1$. Consequently, from Lemma 2.5 and the definition of ρ_k we have

$$\Gamma_k-\|r(\kappa_{k+1})\|\ge\eta_1 pred_k\ge\frac{1}{2}\eta_1\|g_k\|\,\min\Big\{\|d_k\|,\frac{\|g_k\|}{\|H_k\|}\Big\},$$

which implies that

$$\|r(\kappa_{k+1})\|\le\Gamma_k-\frac{\eta_1}{2}\,\|g_k\|\,\min\Big\{\|d_k\|,\frac{\|g_k\|}{\|H_k\|}\Big\}.\tag{41}$$

Second, let $k\in\mathcal{J}$. From Lemma 2.5 we obtain

$$g_k^Td_k\le-\frac{1}{2}\,\|g_k\|\,\min\Big\{\|d_k\|,\frac{\|g_k\|}{\|H_k\|}\Big\}.$$

From Lemma 3.9 and the sufficient decrease condition (30) we have

$$\|r(\kappa_{k+1})\|\le\Gamma_k-\frac{\eta_1}{2}\delta c\,\|g_k\|\min\Big\{\|d_k\|,\frac{\|g_k\|}{\|H_k\|}\Big\}.\tag{42}$$

So, combining (41) and (42) gives

$$\|r(\kappa_{k+1})\|\le\Gamma_k-\gamma\,\|g_k\|\,\min\Big\{\|d_k\|,\frac{\|g_k\|}{\|H_k\|}\Big\},\ \forall\ k\ge0,\tag{43}$$

where $\gamma=\min\{\frac{1}{2}\eta_1,\frac{1}{2}\delta c\}$.

On the other hand, by the definition of Γ_k it holds that

$$\Gamma_{k+1}\le\Gamma_k-\gamma\frac{\|g_k\|}{E_{k+1}}\,\min\Big\{\|d_k\|,\frac{\|g_k\|}{\|H_k\|}\Big\}.$$

But from (28) we have

$$\begin{aligned}E_{k+1}&=1+\sum_{j=0}^{k}\prod_{i=1}^{j}\zeta_{k_i}\\&\le1+\sum_{j=0}^{k}\zeta_{\max}^{j+1}\le\sum_{j=0}^{\infty}\zeta_{\max}^{j}=\frac{1}{1-\zeta_{\max}}.\end{aligned}$$

Hence, $1-\zeta_{\max}\le1/E_{k+1}$. Therefore,

$$(1-\zeta_{\max})\gamma\,\|g_k\|\,\min\Big\{\|d_k\|,\frac{\|g_k\|}{\|H_k\|}\Big\}\le\Gamma_k-\Gamma_{k+1},$$

as required. □

Lemma 3.11. *Let Assumption 3.1(i) holds and $\{\kappa_k\}$ be an infinite sequence generated by Algorithm 2.1. Then the sequence $\{\mu_k\}$ is bounded.*

Proof. Assume that $\{\mu_k\}$ is unbounded. Then there exists a subsequence of $\{\mu_k\}$ such that

$$\mu_k\to+\infty\quad\text{as } k\to+\infty,\tag{44}$$

where without loss any of the generality we denoted the subsequence by $\{\mu_k\}$. From the definition of $ared_k$ and $pred_k$ and using (24) and (36) we obtain

$$\begin{aligned} pred_k &= -g_k^T d_k - \frac{1}{2} d_k^T H_k d_k \\ &= d_k^T (J_k^T J + \mu_k I) d_k - \frac{1}{2} d_k^T H_k d_k \\ &= \frac{1}{2} d_k^T H_k d_k + \mu_k \|d_k\|^2 \\ &\geq \mu_k \|d_k\|^2, \end{aligned} \tag{45}$$

and

$$\begin{aligned} \left|ared_k - pred_k\right| &= \left|\Gamma_k - f(\kappa_k + d_k) + g_k^T d_k + \frac{1}{2} d_k^T J_k^T J_k d_k\right| \\ &\leq \left|\Gamma_k - f(\kappa_k) + O(\|d_k\|)^2\right| \\ &\leq 2f(\kappa_0) + O(\|d_k\|)^2. \end{aligned} \tag{46}$$

Consequently, (45) and (46) imply that

$$\left|\rho_k - 1\right| = \frac{|ared_k - pred_k|}{|pred_k|} \leq \frac{2f(\kappa_0) + O(\|d_k\|)^2}{\mu_k \|d_k\|^2} \to 0 \text{ as } \mu_k \to +\infty.$$

That is $\rho_k \to 1$ as $\mu_k \to +\infty$, which means that $\rho_k \geq \eta_2$ holds for all k sufficiently large. By Step 6 of Algorithm 2.1 there exists $\hat{M} > \bar{\mu}$ such that

$$\beta_k < \hat{M} \quad \text{as } k \to +\infty.$$

Since $f(\kappa)$ is continuously differentiable on $\Omega(\kappa_0)$. Then by Assumption 3.1(i) and Lemma 3.8 $f(\kappa_k)$ is bounded for all $k \geq 0$. But by the structure of μ_k we have μ_k is bounded as $k \to +\infty$ contradicting with (44). □

The following theorem establishes the global convergence of the LM method.

Theorem 3.1. *Let Assumption 3.1 holds and $\{\kappa_k\}$ be generated by Algorithm 2.1. Then*

$$\lim_{k\to\infty} \|g_k\| = \lim_{k\to\infty} \|J_k^T r_k\| = 0. \tag{47}$$

Proof. Assume that (47) does not hold. Then there exists an $\epsilon_0 > 0$ such that

$$\|g_k\| > \epsilon_0 \quad \forall\ k \geq 0. \tag{48}$$

From Lemma 3.11 together with (36) and (48) we have

$$(1 - \zeta_{\max})\gamma\, \epsilon_0\, \min\left\{\|d_k\|, \frac{\epsilon_0}{L^2}\right\} \leq \Gamma_k - \Gamma_{k+1}.$$

But from Lemma 3.8 the sequence $\{\Gamma_k\}$ is non–increasing monotonically and is convergent. Consequently, by summing up the last inequality for all $k \geq 0$ yields

$$(1 - \zeta_{\max})\gamma\, \epsilon_0 \sum_{k=0}^{\infty} \min\left\{\|d_k\|, \frac{\epsilon_0}{L^2}\right\} \leq \sum_{k=0}^{\infty} (\Gamma_k - \Gamma_{k+1}) < +\infty,$$

which implies that

$$\lim_{k\to\infty} \|d_k\| = 0. \tag{49}$$

On the other hand, by Assumption 3.1 and Lemma 3.11 there exists a constant $\nu_3 > 0$ such that

$$\|J_k^T J_k + \mu_k I\| \leq \nu_3 \quad \forall\ k \geq 0.$$

Hence, this together with (24) and (49) give

$$\|g_k\| \leq \|J_k^T J_k + \mu_k I\| \cdot \|d_k\| \leq \nu_3 \|d_k\| \to 0, \text{ as } k \to +\infty,$$

contradicting with (48), which completes the proof. □

4. Local convergence

Let $\mathcal{K}_*$ be the solution set of the problem (10). To establish the local convergence rate it is assumed that the sequence $\{\kappa_k\}$ generated by Algorithm 2.1 converges to a solution $\kappa_* \in \mathcal{K}_*$.

Assumption 4.1. *Assume that for any $\kappa_* \in \mathcal{K}_*$ every $J(\kappa_*)$ has full row rank.*

Lemma 4.12. *Let Assumption 3.1(i) holds. Then there exists an integer $\hat{k} > 0$ such that $\rho_k \geq \eta_2$ for all $k \geq \hat{k}$, where ρ_k is given by (29) and $0 < \eta_2 < 1$ is a given constant.*

Proof. The computed step by (24) is also a solution to the problem

$$\min_{d \in \mathbb{R}^m} \ \widehat{m}_k(d) = \frac{1}{2}\|r(\kappa_k) + J_k d\|^2 + \frac{\mu_k}{2}\|d\|^2. \tag{50}$$

In fact, the first–order optimality conditions of (50) yields (24). Moreover, by Lemma 3.11 the sequence $\{\mu_k\}$ is bounded and since $\partial_B\, r(\kappa_k) \subset \partial_C\, r(\kappa_k)$, then Lemma 1.2 gives

$$\begin{aligned}
\|d_k\|^2 &\leq \frac{2}{\mu_k}\widehat{m}_k(\kappa_* - \kappa_k) \\
&= \frac{1}{\mu_k}\|r(\kappa_k) + J_k(\kappa_* - \kappa_k)\|^2 + \|\kappa_* - \kappa_k\|^2 \\
&= \frac{1}{\mu_k}\|r(\kappa_k) - r(\kappa_*) - J_k(\kappa_k - \kappa_*)\|^2 + \|\kappa_k - \kappa_*\|^2 \\
&= o(\|\kappa_k - \kappa_*\|^2) + \|\kappa_k - \kappa_*\|^2 \ \to 0 \text{ as } k \to +\infty.
\end{aligned}$$

From Lemma 2.6 and using (24) and (35) one has

$$\begin{aligned}
ared_k - \eta_2\, pred_k &= \Gamma_k - f(\kappa_k + d_k) - \eta_2\, pred_k \\
&\geq f(\kappa_k) - f(\kappa_k + d_k) - \eta_2\, pred_k \\
&= -g_k^T d_k - \frac{1}{2} d_k^T J_k^T J_k d_k + o(\|d_k\|^2) - \eta_2\, pred_k \\
&= (1 - \eta_2)\, pred_k + o(\|d_k\|^2) \\
&= \frac{(1 - \eta_2)}{2}\Big[d_k^T J_k^T J_k d_k + 2\mu_k\|d_k\|^2\Big] + o(\|d_k\|^2) \\
&\geq \|d_k\|^2 \Big[\frac{(1 - \eta_2)}{2}\nu_1 + \frac{o(\|d_k\|^2)}{\|d_k\|^2}\Big] \geq 0,
\end{aligned}$$

which implies that there exists an integer $\hat{k} > 0$ such that

$$ared_k \geq \eta_2\, pred_k \quad \forall\ k \geq \hat{k}.$$

Therefore, $\rho_k \geq \eta_2$ for all $k \geq \hat{k}$. □

Next, consider the following lemma; see e.g., [11, Theorem 3.1.4].

Lemma 4.13. *Let A_1 and $A_2 \in \mathbb{R}^{n\times n}$ and A_1 is nonsingular. If $\|A_1^{-1}A_2\| \leq \hat{c} < 1$, then $A_1 + A_2$ is nonsingular and*

$$\|(A_1 + A_2)^{-1}\| \leq \frac{\|A_1^{-1}\|}{1 - \|A_1^{-1}A_2\|}.$$

Lemma 4.14. *Let Assumptions 3.1 and 4.1 hold. Then there exists an integer $\bar{k} > 0$ and a constant $\nu > 0$ such that*

$$\|(H_k + \mu_k I)^{-1}\| \leq 2\nu \quad \forall\ k \geq \bar{k}.$$

Proof. From Assumption 4.1 and Lemma 1.2 there exists a neighborhood

$$\mathcal{N}(\kappa_*, \tau) = \{\kappa \,:\, \|\kappa - \kappa_*\| < \tau\}$$

and a constant $\nu > 0$ such that

$$\|(J(\kappa)^T J(\kappa))^{-1}\| \leq \nu \quad \forall\ J(\kappa) \in \partial_B\, r(\kappa) \quad \forall\ \kappa \in \mathcal{N}(\kappa_*, \tau).$$

In addition, by Step 6 of Algorithm 2.1 and Lemma 4.12 there exists an integer $\hat{k} > 0$ and a constant $M > \bar{\mu} > 0$ such that

$$\mu_k = \beta_k \|r(\kappa_k)\| \leq M\|r(\kappa_k)\| \ \to 0 \quad \forall\ k \geq \hat{k},$$

implying that there exists an integer $\check{k} > 0$ such that

$$\mu_k \leq \frac{1}{2\nu} \quad \forall\ k \geq \check{k}.$$

On the other hand, since $\{\kappa_k\}$ converges to κ_*, then there exists an integer $\tilde{k} > 0$ such that $\kappa_k \in \mathcal{N}(\kappa_*, \tau)$ for all $k \geq \tilde{k}$. So, one has

$$\|\mu_k (J(\kappa)^T J(\kappa))^{-1}\| \leq \frac{1}{2} \quad \forall\ k \geq \bar{k},$$

where $\bar{k} = \max\{\hat{k}, \check{k}, \tilde{k}\}$. By using Lemma 4.13 this gives

$$\|(H_k + \mu_k I)^{-1}\| \leq \frac{\|H_k^{-1}\|}{1 - \|\mu_k H_k^{-1}\|} \leq 2\nu \quad \forall\ k \geq \bar{k}.$$

□

The following theorem establishes the local convergence of the LM method.

Theorem 4.2. *Suppose that both Assumptions 3.1 and 4.1 hold. Moreover, let $\{\kappa_k\}$ be an infinite sequence generated by Algorithm 2.1 and converging to some solution $\kappa_* \in \mathcal{K}_*$. Then the following statements hold:*

(a) *The sequence $\{\kappa_k\}$ converges to κ_* superlinearly, i.e.,*

$$\|\kappa_{k+1} - \kappa_*\| = o(\|\kappa_k - \kappa_*\|).$$

(b) *If $J(\kappa)$ is Lipschitz continuous on $\mathbb{R}^m$, then the sequence $\{\kappa_k\}$ converges to κ_* quadratically, i.e.,*

$$\|\kappa_{k+1} - \kappa_*\| = O(\|\kappa_k - \kappa_*\|^2).$$

Proof. From Lemmas 4.12 and 4.14 and since Assumptions 3.1 and 4.1 hold, then there exists an integer $\bar{k} > 0$ and a constant $\nu > 0$ such that

$$\kappa_{k+1} = \kappa_k + d_k, \quad \|(H_{\bar{k}} + \mu_{\bar{k}} I)^{-1}\| \leq 2\nu \quad \forall\ k \geq \bar{k}.$$

Moreover, Lemma 4.12 and (34) imply that there exists a constant $M > \bar{\mu} > 0$ such that

$$\begin{aligned} \mu_k &= \beta_k \|r(\kappa_k)\| \leq M\|r(\kappa_k)\| \\ &= M\|r(\kappa_k) - r(\kappa_*)\| \\ &\leq ML\|\kappa_k - \kappa_*\| \quad \forall\ k \geq \bar{k}. \end{aligned}$$

Then, from (24) and (36) we have

$$\begin{aligned} \|\kappa_{k+1} - \kappa_*\| &= \|\kappa_k + d_k - \kappa_*\| \\ &= \|\kappa_k - \kappa_* - (H_k + \mu_k I)^{-1} J_k^T r(\kappa_k)\| \\ &\leq \|(H_k + \mu_k I)^{-1}\|\, \|J_k^T r(\kappa_k) - (H_k + \mu_k I)(\kappa_k - \kappa_*)\| \\ &\leq 2\nu \|J_k^T r(\kappa_k) - J_k^T J_k(\kappa_k - \kappa_*) - \mu_k \|\kappa_k - \kappa_*\| \\ &\leq 2\nu \Big[\|J_k\|\, \|r(\kappa_k) - J_k(\kappa_k - \kappa_*)\| + \mu_k \|\kappa_k - \kappa_*\|\Big] \\ &\leq 2\nu\, L\|r(\kappa_k) - r(\kappa_*) - J_k(\kappa_k - \kappa_*)\| + 2ML\nu \|\kappa_k - \kappa_*\|^2. \end{aligned}$$

By Lemma 1.2 and since $\partial_B\, r(\kappa_k) \subset \partial_C\, r(\kappa_k)$, then the above inequality yields

$$\|\kappa_{k+1} - \kappa_*\| = o(\|\kappa_k - \kappa_*\|) + O(\|\kappa_k - \kappa_*\|^2),$$

showing the superlinear rate of the sequence. In addition, if $J(\kappa)$ is Lipschitz continuous on $\mathbb{R}^m$, then by Lemma 1.2 it holds

$$\begin{aligned}\|\kappa_{k+1}-\kappa_*\| &= O(\|\kappa_k-\kappa_*\|^2)+O(\|\kappa_k-\kappa_*\|^2)\\ &= O(\|\kappa_k-\kappa_*\|^2),\end{aligned}$$

showing the quadratic rate of the sequence. □

5. Log–barrier interior–point method for the QEAP

In order to solve the barrier subproblem (13) first and second–order derivatives of the function $\phi^{\hat{\mu}}(\kappa)$ are required, which are given by the following lemma.

Lemma 5.15. *The first and second–order derivatives of the barrier function $\phi^{\hat{\mu}}(\kappa)$ of the subproblem* (13) *are*

$$\begin{aligned}\nabla\phi^{\hat{\mu}}(\kappa) &= \nabla f(\kappa)-\frac{\hat{\mu}}{h(\kappa)}\nabla h(\kappa)\\ &= \nabla f(\kappa)+\frac{2\hat{\mu}}{h(\kappa)}\kappa, \qquad (51)\\ \nabla^2\phi^{\hat{\mu}}(\kappa) &= \nabla^2 f(\kappa)+\frac{\hat{\mu}}{h(\kappa)^2}\nabla h(\kappa)\nabla h(\kappa)^T-\frac{\hat{\mu}}{h(\kappa)}\nabla^2 h(\kappa)\\ &= \nabla^2 f(\kappa)+\frac{4\hat{\mu}}{h(\kappa)^2}\kappa\kappa^T+\frac{2\hat{\mu}}{h(\kappa)}I_m, \qquad (52)\end{aligned}$$

where $\nabla f(\kappa)$ and $\nabla^2 f(\kappa)$ are given by (16) *and* (19)*, respectively, and I_m is the identity matrix.*

Proof. The proof is straightforward and therefore is omitted. □

Newton's method for solving the barrier subproblem (13) at iteration κ_k is based on computing p_k an approximate solution of the equation

$$\nabla^2\phi^{\hat{\mu}}(\kappa_k)p_k=-\nabla\phi^{\hat{\mu}}(\kappa_k). \qquad (53)$$

The optimal Lagrange multiplier of the constrained problem (12) is commonly estimated by

$$z_*\approx\hat{\mu}/h(\kappa), \qquad (54)$$

where the Lagrangian function associated with the constrained problem (12) is defined by

$$\ell(\kappa,z)=f(\kappa)-z\,h(\kappa).$$

By using (51)–(52) the algebraic system of equations (53) is rewritten as

$$\big(\nabla^2 f(\kappa_k)+\frac{z_k}{h(\kappa_k)}\nabla h(\kappa_k)\nabla h(\kappa_k)^T-z_k\nabla^2 h(\kappa_k)\big)p_k=-\big(\nabla f(\kappa_k)-z_k\nabla h(\kappa_k)\big). \qquad (55)$$

A suitable approach to solve (55) is to apply one of the iterative methods GMRES, QMR, or LSQR, which can solve such a system with $\nabla^2\phi^{\hat{\mu}}(\kappa)$ indefinite. The following heuristic, see [19, pp 50], is considered to maintain positive definite Hessian $\mathcal{M}_k:=\nabla^2\phi^{\hat{\mu}}(\kappa_k)$ as follows. Compute the spectral decomposition of $\mathcal{M}_k$:

$$\mathcal{M}_k=Q\Lambda Q^T.$$

Replace $\mathcal{M}_k$ by the perturbed matrix $\mathcal{M}_k+\Delta\mathcal{M}_k=Q(\Lambda+\operatorname{diag}(\tau_i))Q^T$, where $\Delta\mathcal{M}_k$ a correction matrix is such that $\lambda_{\min}(\mathcal{M}_k+\Delta\mathcal{M}_k)\geq\delta$, where $\delta\in(0,1)$ is given and

$$\tau_i=\begin{cases}0, & \lambda_i(\mathcal{M}_k)\geq\delta\\ \delta-\lambda_i, & \lambda_i(\mathcal{M}_k)<\delta.\end{cases}$$

The resulting perturbed Hessian matrix $\mathcal{M}_k+\Delta\mathcal{M}_k$ is strictly positive definite.

For large–scale problems the Hessian $\nabla^2\phi^{\hat{\mu}}(\kappa_k)$ given by (52) is approximated by the following

$$\nabla^2\phi^{\hat{\mu}}(\kappa) \approx \mathcal{H}_k + \frac{4\hat{\mu}}{h(\kappa_k)^2}\kappa_k\kappa_k^T + \frac{2\hat{\mu}}{h(\kappa_k)}I_m, \tag{56}$$

where $\mathcal{H}_k$ is the SR1 quasi–Newton update of $\nabla^2 f(\kappa_k)$ which is given by:

$$\mathcal{H}_{k+1} = \mathcal{H}_k + \frac{(y_k - \mathcal{H}_k s_k)(y_k - \mathcal{H}_k s_k)^T}{(y_k - \mathcal{H}_k s_k)^T s_k}, \tag{57}$$

where $s_k = \kappa_{k+1} - \kappa_k$ and $y_k = \nabla f(\kappa_{k+1}) - \nabla f(\kappa_k)$.

Having obtained an approximate solution p_k of (55), a step–length α_k is computed by a backtracking rule such that the following sufficient decrease condition holds

$$\phi^{\hat{\mu}_k}(\kappa_k + \alpha_k p_k) \leq \phi^{\hat{\mu}_k}(\kappa_k) + \hat{\eta}\alpha_k\nabla\phi^{\hat{\mu}_k}(\kappa_k)^T p_k, \tag{58}$$

where $\hat{\eta} \in (0, \frac{1}{2})$.

The logarithmic barrier Algorithm which solves the barrier subproblem (BP) for a decreasing sequence $\{\hat{\mu}_k\} \downarrow 0$ of the barrier parameter is stated in the following lines.

Algorithm 5.1. (Logarithmic barrier algorithm for solving Problem (CP))
Initialization: Let M, D, N, and B be given constant matrices of the control system (1) and $\hat{\lambda}_i$, $i = 1, \ldots, 2n$ be the desired eigenvalues. Form the matrices A and $\hat{B}$ according to (5).
Specify the parameters $\hat{\mu}_0 > 0$, $a_0 \in (0, 1)$, $\hat{\eta} \in (0, \frac{1}{2})$, and the final tolerance $\epsilon^{\text{tol}} \in (0, 1)$. Create the initial vector κ_0 as explained in Algorithm 2.1 such that $h(\kappa_0) > 0$, and compute J_0 and r_0.
For $k = 0, 1, 2, \ldots,$

(1) Compute a Lagrange multiplier estimate $z_k = \hat{\mu}_k / h(\kappa_k)$.
(2) Compute the approximate solution p_k of (55) by using one of the described methods.
(3) Compute a step–length $\alpha_k > 0$ such that the sufficient decrease condition (58) holds.
(4) Set $\kappa_{k+1} = \kappa_k + \alpha_k p_k$ and compute $r(\kappa_{k+1})$, $J(\kappa_{k+1})$, $h(\kappa_{k+1})$, and $\nabla h(\kappa_{k+1})$.
(5) If $z_k h(\kappa_k) \leq \epsilon^{\text{tol}}$, stop.
(6) Compute $\nabla^2\phi^{\hat{\mu}_k}(\kappa_{k+1})$ by (52) or by (56)–(57), choose $\hat{\mu}_{k+1} \in (0, a_k\hat{\mu}_k)$, set $k := k + 1$ and go to Step 1.

Remark 2. From Algorithm 5.1 the following points are in order:

- The convergence test of Algorithm 5.1 is based on satisfying the complementarity condition $z_* h(\kappa_*) = 0$ at the achieved local solution. A stopping condition based on fulfilling the KKT conditions of the constrained problem (12) is another alternative.
- Instead of leaving the barrier parameter $\hat{\mu}$ fixed until the optimality conditions are satisfied to certain accuracy an adaptive strategy for updating $\hat{\mu}$ at every iteration has shown a nice performance numerically.
- The method switches to compute $\nabla^2\phi^{\hat{\mu}_k}(\kappa)$ using the quasi–Newton update (56)–(57) when the problem size is large.

6. Numerical results

In this section some preliminary testing are given for LM method of Algorithm 2.1 and the logarithmic barrier interior–point method of Algorithm 5.1 denoted by (BM). The two methods are compared with Newton's method (NM) [28]. The methods are implemented using Matlab and all computations are carried out on a Laptop with 2.5 Ghz Core i7–6500 CPU and 8.00 GB RAM.

The following values have been assigned to the parameters of Algorithm 2.1:

$$\eta_1 = 0.1,\ \eta_2 = 0.8,\ \gamma_1 = 0.1,\ \gamma_2 = 10,\ \zeta_{\min} = 10^{-4},$$
$$\zeta_{\max} = 0.9,\ \delta = 0.1,\ \sigma = 0.4,\ \bar{\mu} = 10^{-4},\ \beta_0 = 0.001.$$

Moreover, ζ_k is chosen constant of value $\zeta_k = 0.5$ in the whole implementation. The following values have been assigned to the parameters of Algorithm 5.1:

$$\hat{\mu}_0 = 0.05,\ a_0 = 0.1,\ \hat{\eta} = 10^{-4}.$$

In the following five test problems are considered in details that quite demonstrate the performance of proposed methods. The vector $\hat{\lambda}$ of desired eigenvalues is chosen in each test example according to a specific need. The methods are terminated if the stopping conditions are satisfied, where $\epsilon^{\text{tol}} = 10^{-6}$.

Example 6.1. Consider the quadratic control system (1) with the following data matrices; see [2]:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, D = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}, N = \begin{bmatrix} 10 & -5 \\ -5 & 15 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}.$$

The open–loop system matrix has the eigenvalues $-5, -2.5, \pm 2.2361$i. The desired eigenvalues $\hat{\lambda}_i$'s are chosen as $-5, -2.5, -1.0 \pm 1.0$i, where the first two eigenvalues are left unaltered.

The three methods LM, NM, and BM successfully converged to three different local solutions of the least–squares problem (10) and required 9, 7, and 12 iterations, respectively. The achieved feedback gain matrices by the methods LM and BM are:

$$K_1^{LM} = \begin{bmatrix} -1.8819 & -1.0593 \\ 1.7637 & 1.8819 \end{bmatrix}, \quad K_2^{LM} = \begin{bmatrix} 2.7401 & 1.2262 \\ 0.6597 & -2.7401 \end{bmatrix},$$
$$K_1^{BM} = \begin{bmatrix} -1.7453 & -1.7333 \\ 1.4904 & 1.7453 \end{bmatrix}, \quad K_2^{BM} = \begin{bmatrix} 2.6188 & 2.0294 \\ 0.0065 & -2.6188 \end{bmatrix}.$$

"Table 1 shows the norms of feedback gain matrices achieved by the three methods, along with the results reported in [2].

TABLE 1. Example 6.1: Norms of the achieved feedback controllers

	LM	NM [28]	BM	Method of [2]
$\|K_1\|$	3.3261	3.3579	3.3613	3.1739
$\|K_2\|$	3.1811	3.8871	3.8211	4.7468

Example 6.2. This test problem is from [8], which has the following data matrices:

$$M = 10I_3,\ D = \text{diag}(5, 2.5, 5),$$
$$N = \begin{bmatrix} 1500 & -500 & 0 \\ -500 & 600 & -100 \\ 0 & -100 & 100 \end{bmatrix}, B = \begin{bmatrix} 0.1 & -0.2 \\ 0.2 & -0.3 \\ -0.5 & 0.1 \end{bmatrix}.$$

The open–loop system has the eigenvalues -0.2292 ± 13.1266i, -0.1550 ± 6.3588i, -0.2408 ± 2.6669i, where the desired eigenvalues $\hat{\lambda}_i$'s are chosen as -0.5 ± 10.0i, -0.3 ± 5.0i, -0.4 ± 3.0i.

The methods LM, NM, and BM require 9, 228 and 19 iterations, respectively, to reach three different local solutions. The achieved feedback gain matrices by the methods LM and BM are

$$K_1^{LM} = \begin{bmatrix} -114.4362 & 322.4073 & -231.4457 \\ 394.8490 & 226.2556 & -334.1392 \end{bmatrix}, \quad K_2^{LM} = \begin{bmatrix} 124.5213 & 306.1549 & -72.1854 \\ -343.6139 & -205.3938 & 104.6945 \end{bmatrix},$$
$$K_1^{BM} = \begin{bmatrix} -425.0310 & 178.4601 & -95.6816 \\ 224.9837 & -138.1606 & -489.5446 \end{bmatrix}, \quad K_2^{BM} = \begin{bmatrix} 38.0041 & 149.1344 & -118.0718 \\ -35.2278 & 25.2606 & -17.9726 \end{bmatrix}.$$

Table 2 shows the norms of achieved feedback gain matrices by the three methods together with that obtained by [8].

TABLE 2. Example 6.2: Norms of the feedback controllers

	LM	NM [28]	BM	Method of [8]
$\|K_1\|$	610.9439	572.5650	592.4900	3.7244e+04
$\|K_2\|$	509.4737	224.5839	195.4521	285.3595

Example 6.3. This test problem was considered in [17] and has the following data matrices:

$$M = I_5,\ D = \operatorname{diag}\big(0.2, 0.2\sqrt{3}, 0.4, 0.2\sqrt{3}, 0.2\big),$$
$$N = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix},\ B = \begin{bmatrix} I_3 \\ 0_{2\times 3} \end{bmatrix}.$$

The open–loop system has the eigenvalues -0.1697 ± 1.9219i, -0.1362 ± 1.7244i, -0.1331 ± 1.4080i, -0.1370 ± 0.9919i, -0.1704 ± 0.4895i, where the attempt is to assign the eigenvalues to the desired values -1 ± 3i, -1 ± 2.5i, -1 ± 2i, $-1\pm$i, -1 ± 0.5i. The methods LM, NM, and BM require 16, 7, and 7 iterations to reach three different local solutions. The achieved feedback gain matrices by the methods LM and BM are

$$K_1^{LM} = \begin{bmatrix} -0.2179 & 0.0015 & 0.0367 & -0.0036 & 0.0552 \\ 0.0073 & -0.1764 & -0.0043 & 0.0827 & 0.0008 \\ 0.0325 & 0.0062 & -0.1129 & -0.0070 & 0.0417 \end{bmatrix},$$
$$K_2^{LM} = \begin{bmatrix} -0.0374 & 0.0180 & -0.0009 & 0.0287 & 0.0039 \\ 0.0081 & -0.0387 & 0.0355 & 0.0042 & 0.0037 \\ -0.0006 & 0.0045 & -0.0337 & -0.0202 & -0.0004 \end{bmatrix},$$

$$K_1^{BM} = \begin{bmatrix} -0.2133 & 0.0033 & 0.0378 & -0.0008 & 0.0445 \\ 0.0019 & -0.1689 & 0.0023 & 0.0856 & -0.0042 \\ 0.0360 & -0.0027 & -0.1244 & -0.0005 & 0.0343 \end{bmatrix},$$
$$K_2^{BM} = \begin{bmatrix} -0.0376 & 0.0067 & 0.0026 & 0.0023 & 0.0042 \\ 0.0066 & -0.0381 & 0.0026 & 0.0043 & 0.0098 \\ -0.0024 & 0.0090 & -0.0344 & 0.0219 & -0.0012 \end{bmatrix}.$$

Table 3 shows the norms of achieved feedback gain matrices by the three methods together with that obtained by [17].

TABLE 3. Example 6.3: Norms of the feedback controllers

	LM	NM [28]	BM	Method of [17]
$\|K_1\|$	0.2328	0.2506	0.2290	25.0840
$\|K_2\|$	0.0634	0.1221	0.0475	21.3810

Example 6.4. This test problem is of a discrete shear beam model in multi–input control system (see [14]) with $n = 10$ and $p = 2$. The data matrices are the following

$$M = I_n,\ D = \operatorname{diag}(c_1, \ldots, c_n), N = \begin{bmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & \ldots & 0 & -1 & 2 \end{bmatrix},\ B = \begin{bmatrix} I_p \\ 0_{(n-p)\times p} \end{bmatrix},$$

where $c_k = 0.4\ \sin(k\pi/(n+1))$, $k = 1, \ldots, n$. The open–loop system has 20 eigenvalues, where the attempt is to assign the four eigenvalues $-0.1291 \pm 1.5063\mathrm{i}, -0.1290 \pm 1.3031\mathrm{i}$ of the open–loop system to the desired values $-0.4 \pm 1.5063\mathrm{i}, -0.8 \pm 1.3031\mathrm{i}$ and leave the remaining unchanged. The three methods LM, NM, and BM require 8, 6, and 11 iterations, respectively, to reach a local solution. The feedback matrices obtained by the method BM are:

$$K_{1,BM}^T = \begin{bmatrix} -1.7663 & -0.1880 \\ 0.0809 & -0.1177 \\ 0.6080 & -0.1083 \\ 0.2777 & 0.5609 \\ -0.5359 & 0.1497 \\ -0.4899 & -0.8287 \\ 0.4857 & -0.1122 \\ 0.5287 & 1.0691 \\ -0.3469 & 0.0337 \\ -0.4048 & -1.2030 \end{bmatrix},\ K_{2,BM}^T = \begin{bmatrix} -0.8698 & 0.8900 \\ -0.0405 & -0.1388 \\ 0.0878 & -0.0484 \\ -0.2247 & 0.1302 \\ -0.0747 & -0.0394 \\ 0.4125 & -0.2048 \\ 0.0305 & 0.2007 \\ -0.5038 & 0.2788 \\ -0.0030 & -0.2711 \\ 0.4348 & -0.2102 \end{bmatrix}.$$

Table 4 shows the norms of computed feedback matrices together with that reported by [14].

TABLE 4. Example 6.4: Norms of the feedback controllers

	LM	NM [28]	BM	Method of [14]
$\|K_1\|$	2.9170	2.7894	2.4742	3.3696
$\|K_2\|$	2.2807	1.5643	1.5470	2.3514

Example 6.5. This test problem is of a serially linked mass–spring system with the following data matrices, where $n = 15, p = 2$; see [14, Example 4.5]:

$$M = I_{15},$$

$$D = \begin{bmatrix} -1 & -0.5 & 0 & \ldots & 0 \\ -0.5 & -1 & -0.5 & \ldots & 0 \\ 0 & -0.5 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -0.5 & -0.5 \end{bmatrix},\ N = \begin{bmatrix} 200 & -100 & 0 & \ldots & 0 \\ -100 & 200 & -100 & \ldots & 0 \\ 0 & -100 & 200 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -100 & 100 \end{bmatrix},$$

$$B^T = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 & 2 \\ 0 & 0 & \ldots & 0 & 1 & -3 \end{bmatrix}.$$

The open–loop system has 30 eigenvalues, where the attempt is to assign the four eigenvalues $0.9138 \pm 2.8873\mathrm{i}, 0.9362 \pm 0.3912\mathrm{i}$ of the open–loop system to the desired eigenvalues $-0.4 \pm 0.5171\mathrm{i}, -0.1 \pm 0.2813\mathrm{i}$ and the remaining are kept unchanged. The three methods LM, NM, and BM need 13, 30, and 23 iterations, respectively, to reach three different local solutions of the considered problem. The feedback matrices obtained by the method

BM are:

$$K_{1,BM}^T = \begin{bmatrix} -0.2770 & -0.5059 \\ -0.0552 & -0.9819 \\ 0.5424 & -0.9528 \\ 0.8815 & -0.8384 \\ 0.9492 & -0.7872 \\ 0.9680 & -0.6794 \\ 0.9477 & -0.4588 \\ 0.7939 & -0.1666 \\ 0.4570 & 0.0957 \\ 0.1107 & 0.3657 \\ -0.1666 & 0.6846 \\ -0.4306 & 1.0170 \\ -0.8439 & 1.1669 \\ -1.2680 & 1.0481 \\ -0.7491 & 0.9908 \end{bmatrix}, \quad K_{2,BM}^T = \begin{bmatrix} -0.6888 & 0.5936 \\ -0.5119 & 0.4648 \\ -0.3971 & 0.4093 \\ -0.4285 & 0.5729 \\ -0.4725 & 0.7224 \\ -0.4498 & 0.7552 \\ -0.3287 & 0.6452 \\ -0.1642 & 0.4869 \\ 0.0299 & 0.2769 \\ 0.2177 & 0.0437 \\ 0.4342 & -0.2315 \\ 0.6857 & -0.5132 \\ 0.9476 & -0.7939 \\ 0.9868 & -0.9702 \\ 0.6110 & -0.3186 \end{bmatrix}.$$

Table 5 shows the norms of feedback matrices by the three methods together with that reported by [14].

TABLE 5. Example 6.5: Norms of the achieved feedback controllers

	LM	NM [28]	BM	Method of [14]
$\|K_1\|$	4.7939	3.9081	3.8897	12.5429
$\|K_2\|$	4.8750	2.7145	3.0172	8.4406

The above examples quite show the performance of the considered least–square approach for solving the QEAP. In particular, the method BM successfully achieves feedback gain matrices with smaller norm magnitude compared with those obtained by cited methods. This observation was dominant for almost all considered test problems.

In order to demonstrate the performance of the proposed methods further testing is needed. However, to the best of the author's knowledge no benchmark test problems is available for this problem class. For that purpose 21 test problems were collected from different sources, where the methods LM and BM are compared vs. NM with respect to number of iterations and the CPU–time; see Table 6. The abbreviation N.A. appears in the last column of the table means 'not available'. From the table one sees that sometimes the three methods converge to the same local solution. However, in most cases they converge to three different local solutions. The method NM fails in two instances; in one of them the method fails to reach the stopping condition and in the second the existence of two repeated desired eigenvalues was the reason of failure. In fact, from (19)–(20) the Hessian of f is not defined at points where repeated desired eigenvalues $\hat{\lambda}_i$'s exist. However, repeated desired eigenvalues have no influence on the two methods LM and BM, because the former method skips the second term of the Hessian that causes this difficulty and the second uses a quasi–Newton Hessian update in such a case.

Table 7 summarizes the results of Table 6 and shows the average number of iterations, the average CPU time and number of wins of achieving smaller norm magnitude of computed controllers for the considered 21 test problems. The method LM has the best performance w.r.t. the average number of iterations and CPU time, while the method BM has the best performance of achieving least norm magnitudes of the computed feedback controllers.

CONCLUSION

This work considers the quadratic eigenvalue assignment problem which is described as a nonlinear least–squares problem. Levenberg–Marquardt method that uses a nonmonotone trust region combined with a line search backtracking strategy is proposed to find the local solution of the problem. Moreover, a logarithmic barrier interior–point method

TABLE 6. Comparison between the methods LM, NM, and BM on 21 test problems collected from different sources with respect to number of iterations and computational CPU time.

Source	Problem size		LM		NM [28]		BM		Cited methods	
	n	p	# it.	CPU	# it.	CPU	# it.	CPU		
			$\|K_1\|$	$\|K_2\|$	$\|K_1\|$	$\|K_2\|$	$\|K_1\|$	$\|K_2\|$	$\|K_1\|$	$\|K_2\|$
[13]	2	1	8	0.19	8	0.14	11	0.13		
			4.50	9.63	4.50	9.63	4.50	9.63	N.A.	N.A.
[25]	2	1	5	0.05	6	0.08	5	0.08		
			5.99	8.62	5.99	8.62	5.97	5.68	6.09	8.05
[2]	2	2	9	0.08	6	0.09	12	0.03		
			3.32	3.18	3.35	3.89	3.36	3.82	3.17	4.75
[22]	3	1	10	0.28	15	0.14	13	0.14		
			0.96	0.65	0.97	0.64	0.96	0.65	N.A.	N.A.
[17]	3	2	9	0.09	14	0.09	21	0.34		
			23.40	21.65	23.99	18.61	27.01	18.80	162.54	109.88
[10]	4	2	11	0.09	15	0.13	33	0.50		
			0.25	0.19	0.42	0.49	0.20	0.13	0.99	0.99
[3]	3	2	11	0.09	19	0.14	22	0.24		
			26.43	24.60	25.55	22.37	27.14	18.53	61.43	8.67
[8]	3	2	9	0.06	228	0.25	19	0.17		
			610.9	509.4	572.6	224.6	592.6	195.2	3.7e4	285.4
[16]	4	1	9	0.08	20	0.09	32	0.27		
			19.48	9.74	19.57	9.74	19.49	9.74	N.A.	N.A.
[8]	5	2	6	0.17	7	0.17	12	0.67		
			4.01	2.07	4.03	2.08	3.67	2.21	84.79	43.39
[12]	5	2	10	0.21	F	F	35	2.11		
			0.16	1.54	–	–	0.22	1.94	0.70	0.37
[15]	3	2	6	0.06	9	0.14	10	0.13		
			1.56	0.82	1.51	0.74	1.47	0.66	11.62	2.01
[4]	4	2	6	0.06	6	0.08	6	0.14		
			3.58	1.98	3.62	1.95	3.58	1.70	N.A.	N.A.
[5]	4	2	3	0.06	3	0.05	5	0.13		
			0.05	0.01	0.05	0.01	0.05	0.01	N.A.	N.A.
[17]	5	3	9	0.11	7	0.16	7	0.6		
			0.23	0.06	0.25	0.12	0.22	0.05	25.08	21.38
[2]	10	2	8	0.19	7	0.19	48	0.38		
			2.91	2.28	2.78	1.56	3.0	1.24	3.60	2.71
[26]	10	3	7	0.27	6	0.20	19	0.38		
			0.44	0.35	0.49	0.32	0.46	0.35	0.69	0.17
[14]	10	2	8	0.25	17	3.56	11	3.77		
			2.91	2.28	2.78	1.56	2.47	1.54	3.60	2.72
[14]	15	2	13	0.36	20	0.33	112	1.60		
			4.87	4.97	3.99	3.71	3.96	3.64	8.44	4.02
[3]	50	2	11	2.70	49	2.60	70	3.90		
			1.41	1.98	1.26	1.74	0.59	0.60	3.41	3.41
[14]	100	2	22	8.41	F	F	70	8.47		
			2.45	3.44	–	–	1.67	1.53	6.79	6.82

TABLE 7. Comparison between the methods LM and BM vs. NM on the considered 21 test problems with respect to average number of iterations and average computational CPU time.

	LM	NM [28]	BM
Average No. of iterations	9	44	27
Average CPU time (sec.)	0.66	0.75	1.12
No. of wins of smaller $\|K_1\|$ & $\|K_2\|$	7	6	19

is addressed to tackle an inequality constrained problem resulting from incorporating an upper bound on the computed feedback controllers.

The considered approach is characterized by the following features:

- The least–squares problem formulation lies in the category of partial eigenvalue assignment, where the problem structure allows to assign part or all eigenvalues of the system.
- Levenberg–Marquardt method ideally cope with the considered problem. As the control system is converted from second into first order, the problem size is doubled. However, the method has shown a good performance and was capable to solve moderate large problems in a few seconds.
- Numerically, the proposed Levenberg–Marquardt method outperformed Newton's method [28] with respect to number of iterations and the CPU time on 21 test problems collected from different sources.
- Over the considered set of test problems the logarithmic barrier interior–point method successfully achieved feedback controllers with smaller norm magnitude than those obtained by their cited counterparts.

Conflict of interest

The authors declare that they have no conflict of interest regarding the publication of this article.

References

[1] Z.J. Bai, B.N. Datta and J. Wang. Robust and minimum norm partial quadratic eigenvalue assignment in vibrating systems: A new optimization approach, *Mechanical Systems and Signal Processing*, **24** (2010), 766-–783.

[2] Z.J. Bai, M. Lu and Q.Y. Wan. Minimum norm partial quadratic eigenvalue assignment for vibrating structures using receptances and system matrices, *Mechanical Systems and Signal Processing*, **112** (2018), 265-–279.

[3] Z.J. Bai, J.K. Yang and B.N. Datta. Robust partial quadratic eigenvalue assignment with time delay using the receptance and the system matrices, *Journal of Sound and Vibration*, **384** (2016), 1–14.

[4] S. Brahma and B. Datta. An optimization approach for minimum norm and robust partial quadratic eigenvalue assignment problems for vibrating structures, *Journal of Sound and Vibration*, **324** (2009), 471—489.

[5] Y.F. Cai, J. Qian and S.F. Xu. The formulation and numerical method for partial quadratic eigenvalue assignment problems, *Numerical Linear Algebra with Applications*, **18**:4 (2011), 637-–652.

[6] L. Chen. A modified Levenberg–Marquardt method with line search for nonlinear equations, *Computational Optimization and Applications*, **65** (2016), 753-–779.

[7] L. Chen and Y. Ma. A modified Levenberg–Marquardt method for solving system of nonlinear equations, *Journal of Applied Mathematics and Computing*, **69** (2023), 2019-–2040.

[8] E.K. Chu. Pole assignment for second–order systems, *Mechanical Systems and Signal Processing*, **16**:1 (2002), 39-–59.

[9] F.H. Clarke. *Optimization and nonsmooth analysis*, John Wiley, New York, 1983.

[10] B.N. Datta, S. Elhay, Y.M. Ram and D.R. Sarkissian. Partial eigenstructure for the quadratic pencil, *Journal of Sound and Vibration*, **230**:1 (2000), 101–110.

[11] J.E. Dennis and R.B. Schnabel. *Numerical methods for unconstrained optimization and nonlinear equations*, SIAM, Philadelphia, 1996.

[12] D. Henrion, M. Sebek and V. Kučera. Robust pole placement for second–order systems: an LMI approach. *Kybernetika*, **41** (2005), 1-–14.

[13] S.S. Kandala, S. Chakraborty, T.K. Uchida and C.P. Vyasarayani. Hybrid method of receptances and optimization–based technique for pole placement in time–delayed systems, *International Journal of Dynamics and Control*, **8** (2020), 558-–569.

[14] H. Liu, B.X. He and X.P. Chen. Minimum norm partial quadratic eigenvalue assignment for vibrating structures using receptance method, *Mechanical Systems and Signal Processing*, **123** (2019), 131-–142.
[15] H. Liu and Y. Yuan. A multi–step method for partial quadratic pole assignment problem with time delay, *Applied Mathematics and Computation*, 283 (2016), 29-–35.
[16] H. Liu, R. Li and Y. Din. Partial eigenvalue assignment for gyroscopic second–order systems with time delay, *Mathematics*, **8** (2020), 1235.
[17] M. Lu and H. Xie. A method for robust quadratic pole assignment, *Transactions of the Institute of Measurement and Control*, **44**:14 (2022), 2756—2767.
[18] E.M.E. Mostafa, M.A. Tawhid and E.R. Elwan. Inexact Newton and quasi–Newton methods for the output feedback pole assignment problem, *Computational and Applied Mathematics*, **33** (2014), 517–542.
[19] J. Nocedal and S.J. Wright. *Numerical Optimization*. Springer Series in Operations Research, Springer, 2006.
[20] L. Qi and J. Sun. A nonsmooth version of Newton's method, *Mathematical Programming*, **58** (1993), 353–367.
[21] L. Qi, X. Xiao and L. Zhang. A parameter–self adjusting Levenberg–Marquardt method for solving nonsmooth equations, *Journal of Computational Mathematics*, **34**:3 (2016), 317–338.
[22] R.G. Salsa and F. Ma. *Advances in the theory of system decoupling*, Springer, 2021.
[23] D. Sun and J. Sun. Strong semismoothness of eigenvalues of symmetric matrices and its application to inverse eigenvalue problems, *SIAM Journal of Numerical Analysis*, **40** (2003), 2352–2367.
[24] F. Tisseur and K. Meerbergen. The quadratic eigenvalue problem, *SIAM Review*, **43**:2 (2001), 235-–286.
[25] X. Wei, R. Xia, J.E. Mottershead, and X. He. H_2–optimal vibration control using receptance–based regional eigenvalue assignment, *Mechanical Systems and Signal Processing*, **167** (2022), 108591.
[26] H. Xie. A receptance method for robust and minimum norm partial quadratic eigenvalue assignment, *Mechanical Systems and Signal Processing*, **160** (2021), 107838.
[27] H. Xie and K. Hu. Schur method for robust partial quadratic eigenvalue assignment problem, *Computational and Applied Mathematics*, **44**:1 (2025), Article 41.
[28] K. Yang and R. Orsi. Static output feedback pole assignment via a trust region approach, *IEEE Transactions on Automatic Control*, **52** (2007), 2146–2150.
[29] Y. Yang and L. Qi. Smoothing trust region methods for nonlinear complementarity problems with P_0–function. *Annals of Operations Research*, **133** (2005), 99—117.
[30] K. Zhao, F. Deng and Z. Liu. Partial quadratic eigenvalue assignment for vibration systems using receptances and system matrices, *Journal of Sound and Vibration*, **575**, (2024), 118245.
[31] W. Zhou. On the convergence of the modified Levenberg-–Marquardt method with a non-monotone second order Armijo type line search, *Journal of Computational and Applied Mathematics*, **239** (2013), 152-–161.

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