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A COLLECTION OF INEQUALITIES INVOLVING THE LOGARITHMIC FUNCTION

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Abstract. The main topic of this article is the derivation of interpolation inequalities involving logarithmic functions. There are eight inequalities presented in this manuscript which involve logarithmic functions of a single real variable. The results are novel and the approach taken to obtain the results relies purely on functional inequalities and not on monotonicity properties or series expansions as other works in the literature. Logarithmic functions are ubiquitous in Mathematics, and one could think of Analytic Number Theory where these class of functions appears a lot. It is a contemporary research area to find appropriate estimates, and this work was strongly motivated and intensely infuenced by the works of Bagul-Chesneau and Kostic. This article demonstrates the elegance of integral inequalities, and using these integral inequalities in a smart way enables to generate logarithmic inequalities avoiding the computational challenges that the monotonicity approach imposes. All the results have been rigorously proved theoretically and there are graphical demonstrations that verify the theory behind the obtained inequalities.

1. Introduction

In this work, some novel interpolation inequalities are presented involving logarithmic functions. Logarithmic functions are ubiquitous in various branches of mathematics (i.e. analytic number theory [1], [6], [12], [9]) and being able to find proper estimates involving these functions is a contemporary research area [2], [3],[4], [5], [10], [11], [14], [13]. In order to prove the results in this article, there is reliance only on integral inequalities, avoiding to use monotonicity techniques as it is the most dominant technique in the literature. The approach taken fully diverges from what exists in the literature, using only Schwarz-Cauchy, H*ölder*, and Chebyshev integral inequalities. The inequalities derived are original, and to the knowledge of the author such inequalities do not appear in the literature, comparing with classical works of [7],[8],[15], [16]. The motivation for the author to produce this work comes from the works of Chesneau-Bagul [3], and Kostic [10]. These

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works had a positive impact on the author to produce new estimates involving logarithmic functions. More precisely, inequality (2) is a sharp bound for the logarithmic function and simpler compared to the Chesneau-Bagul bound. The other estimates are non-trivial and serve as an excellent enrichment on the literature of logarithmic inequalities. The article is organized in a particular manner. Firstly, the main results are presented, then rigorous proofs of the inequalities follow in a separate section. There is a third section, where the inequalities are graphically represented. Last but not least, there is a section of conclusions and remarks.

2. Main Results

Theorem 2.1.

$$\ln(x+1) \leq 5^{\frac{5}{6}} x^{\frac{1}{6}} \left(1-(x+1)^{-\frac{1}{5}}\right)^{\frac{5}{6}} \ (x+1)^{-2} \left(\frac{1}{3}(x+1)^3 + x + 1\right)^{\frac{1}{2}} (\arctan(x+1))^{\frac{1}{2}} \times (\exp(x+1)+x)^{\frac{1}{2}} (x+1-\ln(\exp(x+1)+1)+\ln(2))^{\frac{1}{2}}\ , \ \forall\, x \ \in [0,+\infty[. \tag{1}$$

Theorem 2.2.

$$\ln(x+1) \leq \ x^{\frac{1}{17}} 16^{\frac{16}{17}} \left(1-(x+1)^{-\frac{1}{16}}\right)^{\frac{16}{17}}, \ \forall\, x \ \in [0,+\infty[. \tag{2}$$

Theorem 2.3.

$$x \leq \frac{1}{3}\ (\ln(x+1))^{\frac{1}{20}} \left(\frac{19}{20}\right)^{\frac{19}{20}} \left((x+1)^{\frac{20}{19}}-1\right)^{\frac{19}{20}} + \frac{1}{3}\ (\ln(x+1))^{\frac{1}{21}} \left(\frac{20}{21}\right)^{\frac{20}{21}} \left((x+1)^{\frac{21}{20}}-1\right)^{\frac{20}{21}} + \frac{1}{3}\ (\ln(x+1))^{\frac{1}{22}} \left(\frac{21}{22}\right)^{\frac{21}{22}} \left((x+1)^{\frac{22}{21}}-1\right)^{\frac{21}{22}}, \quad \forall\, x \ \in [0,+\infty[. \tag{3}$$

Theorem 2.4.

$$\begin{aligned} |\ln(x+1)(x+1)-x| \ \leq\ & \frac{1}{3}\left(\frac{1}{3}\right)^{\frac{1}{2}} (\ln(x+1))^{\frac{3}{2}} \ \left(\frac{1}{2}\right)^{\frac{1}{2}} \left((x+1)^2-1\right)^{\frac{1}{2}} \\ & + \frac{1}{3}\left(\frac{1}{4}\right)^{\frac{1}{3}} (\ln(x+1))^{\frac{4}{3}} \left(\frac{2}{3}\right)^{\frac{2}{3}} \left((x+1)^{\frac{3}{2}}-1\right)^{\frac{2}{3}} \\ & + \frac{1}{3}\ \left(\frac{1}{6}\right)^{\frac{1}{5}} (\ln(x+1))^{\frac{6}{5}} \left(\frac{4}{5}\right)^{\frac{4}{5}} \left((x+1)^{\frac{5}{4}}-1\right)^{\frac{4}{5}}, \ \forall\, x \ \in [0,+\infty[. \end{aligned} \tag{4}$$

Theorem 2.5.

$$|\ln(x+1)(x+1)-x|^{\ 3} \leq (\ln(x+1))^{\frac{9}{2}} \ \left(\frac{1}{6}\right)^{\frac{3}{2}} \left(x^2+2x\right)^{\frac{3}{2}}, \ \forall\, x \ \in [0,+\infty[. \tag{5}$$

Theorem 2.6.

$$\ln(\ln(x+1)+1) \leq \frac{\ln(x+1)}{1+\ln(x+1)} + \left(\frac{1}{16}\right)^{\frac{1}{3}} \ (\ln(x+1))^{\frac{4}{3}} \left(1-(\ln(x+1)+1)^{-2}\right)^{\frac{2}{3}}, \ \forall\, x \ \in [0,+\infty[. \tag{6}$$

Theorem 2.7.

$$
\begin{aligned}
&\frac{1}{2}\ln\left(\left(2x^2+1\right)\left(3x^2+1\right)\left(5x^2+1\right)\left(7x^2+1\right)\right)\\
&\leq 2^{\frac{-1}{4}}\sqrt{\arctan\left(\sqrt{2}x\right)}\,\sqrt{2x-2^{\frac{1}{2}}\arctan\left(\sqrt{2}x\right)}\\
&+3^{\frac{-1}{4}}\sqrt{\arctan\left(\sqrt{3}x\right)}\,\sqrt{3x-3^{\frac{1}{2}}\arctan\left(\sqrt{3}x\right)}\\
&+5^{\frac{-1}{4}}\sqrt{\arctan\left(\sqrt{5}x\right)}\,\sqrt{5x-5^{\frac{1}{2}}\arctan\left(\sqrt{5}x\right)}\\
&+7^{\frac{-1}{4}}\sqrt{\arctan\left(\sqrt{7}x\right)}\,\sqrt{7x-7^{\frac{1}{2}}\arctan\left(\sqrt{7}x\right)}
\end{aligned}
\tag{7}
$$

$x \in [0, +\infty[$.

Theorem 2.8.

$$
x^2\ln\left(x^2+1\right) \geq 2x(x-\arctan(x)),\ \ x\ \in [0,+\infty[. \tag{8}
$$

3. Rigorous proofs of the main results

In this section, all the inequalities are derived using rigorous proofs.

3.1. **Proof of Theorem 1.**

Proof. Let $I(x) = \ln(x+1)(x+1)^2, D_I = [0, +\infty[$. Then it follows that

$$\begin{aligned}
I(x) &= \ln(x+1)(x+1)^2 \\
&= \int_0^x \frac{1}{t+1} dt \int_0^{x+1} \frac{\sqrt{a(t)+1}}{\sqrt{a(t)+1}} dt \int_0^{x+1} \frac{\sqrt{b(t)+1}}{\sqrt{b(t)+1}} dt \\
&(a(t) = t^2 > 0, b(t) = \exp(t) > 0) \\
&= \int_0^x \frac{1}{t+1} dt \int_0^{x+1} \sqrt{a(t)+1} \frac{1}{\sqrt{a(t)+1}} dt \int_0^{x+1} \sqrt{b(t)+1} \frac{1}{\sqrt{b(t)+1}} dt \\
\textit{Hölder and Schwarz} &\leq \left(\int_0^x dt\right)^{\frac{1}{p}} \left(\int_0^x \left((t+1)^{-1}\right)^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}} \left(\int_0^{x+1} \left(\sqrt{a(t)+1}\right)^2 dt\right)^{\frac{1}{2}} \\
&\times \left(\int_0^{x+1} \left(\frac{1}{\sqrt{a(t)+1}}\right)^2 dt\right)^{\frac{1}{2}} \left(\int_0^{x+1} \left(\sqrt{b(t)+1}\right)^2 dt\right)^{\frac{1}{2}} \\
&\times \left(\int_0^{x+1} \left(\frac{1}{\sqrt{b(t)+1}}\right)^2 dt\right)^{\frac{1}{2}} \\
&= x^{\frac{1}{p}} (p-1)^{\frac{p-1}{p}} \left(1 - (x+1)^{-\frac{1}{p-1}}\right)^{\frac{p-1}{p}} \\
&\times \left(\int_0^{x+1} (a(t)+1) dt\right)^{\frac{1}{2}} \left(\int_0^{x+1} \left(\frac{1}{a(t)+1}\right) dt\right)^{\frac{1}{2}} \\
&\times \left(\int_0^{x+1} (b(t)+1) dt\right)^{\frac{1}{2}} \left(\int_0^{x+1} \left(\frac{1}{b(t)+1}\right) dt\right)^{\frac{1}{2}} \\
&= x^{\frac{1}{p}} (p-1)^{\frac{p-1}{p}} \left(1 - (x+1)^{-\frac{1}{p-1}}\right)^{\frac{p-1}{p}} \\
&\times \left(\frac{1}{3}(x+1)^3 + x + 1\right)^{\frac{1}{2}} (\arctan(x+1))^{\frac{1}{2}} \\
&\times (\exp(x+1) + x)^{\frac{1}{2}} (x + 1 - \ln(\exp(x+1)+1) + \ln(2))^{\frac{1}{2}}.
\end{aligned}$$

By choosing $p = 6$ and dividing both parts of the inequality by $(x+1)^2$, this gives

$$\begin{aligned}
\ln(x+1) &\leq 5^{\frac{5}{6}} x^{\frac{1}{6}} \left(1 - (x+1)^{-\frac{1}{5}}\right)^{\frac{5}{6}} (x+1)^{-2} \left(\frac{1}{3}(x+1)^3 + x + 1\right)^{\frac{1}{2}} (\arctan(x+1))^{\frac{1}{2}} \\
&\times (\exp(x+1) + x)^{\frac{1}{2}} (x + 1 - \ln(\exp(x+1)+1) + \ln(2))^{\frac{1}{2}}, \ \forall\, x \in [0, +\infty[.
\end{aligned}$$

□

3.2. **Proof of Theorem 2.**

Proof. From fundamental theorem of Calculus, $\ln(x+1) = \int_0^x \frac{1}{t+1}dt,\ t \in (0,x), x \in [0,+\infty[$. Consequently, this yields

$$\begin{aligned}
\ln(x+1) &= \int_0^x \frac{1}{t+1}dt \\
&= \int_0^x (t+1)^{-1}dt \\
H\ddot{o}lder's\ inequality &\leq \left(\int_0^x dt\right)^{\frac{1}{p}} \left(\int_0^x \left((t+1)^{-1}\right)^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}} \\
&= \left(\int_0^x dt\right)^{\frac{1}{p}} \left(\int_0^x (t+1)^{-\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}} \\
&= x^{\frac{1}{p}}\ (p-1)^{\frac{p-1}{p}} \left(1-(x+1)^{-\frac{1}{p-1}}\right)^{\frac{p-1}{p}},\ p \in]1,+\infty[. \\
\Rightarrow \ln(x+1) &\leq x^{\frac{1}{p}}\ (p-1)^{\frac{p-1}{p}} \left(1-(x+1)^{-\frac{1}{p-1}}\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Choosing $p = 17$, the above inequality takes the form

$$\ln(x+1) \leq\ x^{\frac{1}{17}} 16^{\frac{16}{17}} \left(1-(x+1)^{-\frac{1}{16}}\right)^{\frac{16}{17}},\ \forall\ x\ \in [0,+\infty[$$

and the proof is complete. □

3.3. **Proof of Theorem 3.**

Proof. Let $f(t) = \exp(t)$ and $d(x) = \ln(x+1)$, $t \in (0, d(x)), x \in (0,+\infty)$. Then

$$\begin{aligned}
&\Big|\int_0^{d(x)} f(t)dt + \int_0^{d(x)} f'(t)dt + \int_0^{d(x)} f''(t)dt\Big| \\
&\leq \Big|\int_0^{d(x)} f(t)dt\Big| + \Big|\int_0^{d(x)} f'(t)dt\Big| + \Big|\int_0^{d(x)} f''(t)dt\Big| \\
&\leq \int_0^{d(x)} |f(t)|dt + \int_0^{d(x)} |f'(t)|dt + \int_0^{d(x)} |f''(t)|dt \\
&\leq \left(\int_0^{d(x)} dt\right)^{\frac{1}{p}} \left(\int_0^{d(x)} |f(t)|^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}} + \left(\int_0^{d(x)} dt\right)^{\frac{1}{q}} \left(\int_0^{d(x)} |f'(t)|^{\frac{q}{q-1}} dt\right)^{\frac{q-1}{q}} \\
&+ \left(\int_0^{d(x)} dt\right)^{\frac{1}{r}} \left(\int_0^{d(x)} |f''(t)|^{\frac{r}{r-1}} dt\right)^{\frac{r-1}{r}} \\
&= (d(x))^{\frac{1}{p}} \left(\int_0^{d(x)} |f(t)|^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}} + (d(x))^{\frac{1}{q}} \left(\int_0^{d(x)} |f'(t)|^{\frac{q}{q-1}} dt\right)^{\frac{q-1}{q}} \\
&+ (d(x))^{\frac{1}{r}} \left(\int_0^{d(x)} |f''(t)|^{\frac{r}{r-1}} dt\right)^{\frac{r-1}{r}} \\
&= (\ln(x+1))^{\frac{1}{p}} \left(\frac{p-1}{p}\right)^{\frac{p-1}{p}} \left((x+1)^{\frac{p}{p-1}} - 1\right)^{\frac{p-1}{p}} \\
&+ (\ln(x+1))^{\frac{1}{q}} \left(\frac{q-1}{q}\right)^{\frac{q-1}{q}} \left((x+1)^{\frac{q}{q-1}} - 1\right)^{\frac{q-1}{q}} \\
&+ (\ln(x+1))^{\frac{1}{r}} \left(\frac{r-1}{r}\right)^{\frac{r-1}{r}} \left((x+1)^{\frac{r}{r-1}} - 1\right)^{\frac{r-1}{r}}
\end{aligned}$$

by employing the triangle inequality and Hölder's inequality. Consequently, the inequality obtained is

$$\begin{aligned}3x \leq (\ln(x+1))^{\frac{1}{p}} \left(\frac{p-1}{p}\right)^{\frac{p-1}{p}} \left((x+1)^{\frac{p}{p-1}}-1\right)^{\frac{p-1}{p}}\\ +(\ln(x+1))^{\frac{1}{q}} \left(\frac{q-1}{q}\right)^{\frac{q-1}{q}} \left((x+1)^{\frac{q}{q-1}}-1\right)^{\frac{q-1}{q}}\\ +(\ln(x+1))^{\frac{1}{r}} \left(\frac{r-1}{r}\right)^{\frac{r-1}{r}} \left((x+1)^{\frac{r}{r-1}}-1\right)^{\frac{r-1}{r}}.\end{aligned}$$

Dividing both sides by 3 yields

$$\begin{aligned}x \leq \frac{1}{3}(\ln(x+1))^{\frac{1}{p}} \left(\frac{p-1}{p}\right)^{\frac{p-1}{p}} \left((x+1)^{\frac{p}{p-1}}-1\right)^{\frac{p-1}{p}}\\ +\frac{1}{3}(\ln(x+1))^{\frac{1}{q}} \left(\frac{q-1}{q}\right)^{\frac{q-1}{q}} \left((x+1)^{\frac{q}{q-1}}-1\right)^{\frac{q-1}{q}}\\ +\frac{1}{3}(\ln(x+1))^{\frac{1}{r}} \left(\frac{r-1}{r}\right)^{\frac{r-1}{r}} \left((x+1)^{\frac{r}{r-1}}-1\right)^{\frac{r-1}{r}}.\end{aligned}$$

Chosing Hölder exponents $p = 20$, $q = 21$, $r = 22$ yields the final result

$$\begin{aligned}x \leq \frac{1}{3}\ (\ln(x+1))^{\frac{1}{20}} \left(\frac{19}{20}\right)^{\frac{19}{20}} \left((x+1)^{\frac{20}{19}}-1\right)^{\frac{19}{20}} + \frac{1}{3}\ (\ln(x+1))^{\frac{1}{21}} \left(\frac{20}{21}\right)^{\frac{20}{21}} \left((x+1)^{\frac{21}{20}}-1\right)^{\frac{20}{21}}\\ +\frac{1}{3}\ (\ln(x+1))^{\frac{1}{22}} \left(\frac{21}{22}\right)^{\frac{21}{22}} \left((x+1)^{\frac{22}{21}}-1\right)^{\frac{21}{22}},\ \ \forall\ x\ \in [0, +\infty[.\end{aligned}$$

□

3.4. **Proof of Theorem 4.**

Proof. Let $f(t) = \exp(t)$ and $d(x) = \ln(x+1)$, $t \in (0, d(x))$, $x \in (0, +\infty)$. Then

$$\begin{aligned}
&\Big|\int_0^{d(x)} tf(t)dt + \int_0^{d(x)} tf'(t)dt + \int_0^{d(x)} tf''(t)dt\Big| \\
&\leq \Big|\int_0^{d(x)} tf(t)dt\Big| + \Big|\int_0^{d(x)} tf'(t)dt\Big| + \Big|\int_0^{d(x)} tf''(t)dt\Big| \\
&\leq \int_0^{d(x)} |tf(t)|dt + \int_0^{d(x)} |tf'(t)|dt + \int_0^{d(x)} |tf''(t)|dt \\
&\leq \left(\int_0^{d(x)} t^p dt\right)^{\frac{1}{p}} \left(\int_0^{d(x)} |f|^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}} \\
&+ \left(\int_0^{d(x)} t^q dt\right)^{\frac{1}{q}} \left(\int_0^{d(x)} |f'|^{\frac{q}{q-1}} dt\right)^{\frac{q-1}{q}} \\
&+ \left(\int_0^{d(x)} t^r dt\right)^{\frac{1}{r}} \left(\int_0^{d(x)} |f''|^{\frac{r}{r-1}} dt\right)^{\frac{r-1}{r}} \\
&= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{p-1}{p}\right)^{\frac{p-1}{p}} (\ln(x+1))^{\frac{p+1}{p}} \left((x+1)^{\frac{p}{p-1}} - 1\right)^{\frac{p-1}{p}} \\
&+ \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\frac{q-1}{q}\right)^{\frac{q-1}{q}} (\ln(x+1))^{\frac{q+1}{q}} \left((x+1)^{\frac{q}{q-1}} - 1\right)^{\frac{q-1}{q}} \\
&+ \left(\frac{1}{r+1}\right)^{\frac{1}{r}} \left(\frac{r-1}{r}\right)^{\frac{r-1}{r}} (\ln(x+1))^{\frac{r+1}{r}} \left((x+1)^{\frac{r}{r-1}} - 1\right)^{\frac{r-1}{r}}.
\end{aligned}$$

by employing the triangle inequality and Hölder's inequality. The inequality takes the form

$$\begin{aligned}
|\ln(x+1)(x+1) - x| \leq\ & \frac{1}{3}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{p-1}{p}\right)^{\frac{p-1}{p}} (\ln(x+1))^{\frac{p+1}{p}} \left((x+1)^{\frac{p}{p-1}} - 1\right)^{\frac{p-1}{p}} \\
&+ \frac{1}{3}\left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\frac{q-1}{q}\right)^{\frac{q-1}{q}} (\ln(x+1))^{\frac{q+1}{q}} \left((x+1)^{\frac{q}{q-1}} - 1\right)^{\frac{q-1}{q}} \\
&+ \frac{1}{3}\left(\frac{1}{r+1}\right)^{\frac{1}{r}} \left(\frac{r-1}{r}\right)^{\frac{r-1}{r}} (\ln(x+1))^{\frac{r+1}{r}} \left((x+1)^{\frac{r}{r-1}} - 1\right)^{\frac{r-1}{r}}.
\end{aligned}$$

Choosing $p = 2, q = 3, r = 5$, the final estimate is obtained

$$\begin{aligned}
|\ln(x+1)(x+1) - x| \ \leq\ & \frac{1}{3}\left(\frac{1}{3}\right)^{\frac{1}{2}} (\ln(x+1))^{\frac{3}{2}} \left(\frac{1}{2}\right)^{\frac{1}{2}} \left((x+1)^2 - 1\right)^{\frac{1}{2}} \\
&+ \frac{1}{3}\left(\frac{1}{4}\right)^{\frac{1}{3}} (\ln(x+1))^{\frac{4}{3}} \left(\frac{2}{3}\right)^{\frac{2}{3}} \left((x+1)^{\frac{3}{2}} - 1\right)^{\frac{2}{3}} \\
&+ \frac{1}{3}\left(\frac{1}{6}\right)^{\frac{1}{5}} (\ln(x+1))^{\frac{6}{5}} \left(\frac{4}{5}\right)^{\frac{4}{5}} \left((x+1)^{\frac{5}{4}} - 1\right)^{\frac{4}{5}},\ \forall\ x\ \in [0, +\infty[.
\end{aligned}$$

□

4. Proof of Theorem 5

Proof. Let $f(t) = \exp(t)$ and $d(x) = \ln(x+1)$, $t \in (0, d(x)), x \in (0, +\infty)$. Then

$$
\begin{aligned}
&\Big|\int_0^{d(x)} tf(t)dt \int_0^{d(x)} tf'(t)dt \int_0^{d(x)} tf''(t)dt\Big| \\
&= \Big|\int_0^{d(x)} tf(t)dt\Big| \; \Big|\int_0^{d(x)} tf'(t)dt\Big| \; \Big|\int_0^{d(x)} tf''(t)dt\Big| \\
&\leq \int_0^{d(x)} |tf(t)|dt \int_0^{d(x)} |tf'(t)|dt \int_0^{d(x)} |tf''(t)|dt \\
&= \left(\int_0^{d(x)} t^p dt\right)^{\frac{1}{p}} \left(\int_0^{d(x)} |f|^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}} \\
&\times \left(\int_0^{d(x)} t^q dt\right)^{\frac{1}{q}} \left(\int_0^{d(x)} |f'|^{\frac{q}{q-1}} dt\right)^{\frac{q-1}{q}} \\
&\times \left(\int_0^{d(x)} t^r dt\right)^{\frac{1}{r}} \left(\int_0^{d(x)} |f''|^{\frac{r}{r-1}} dt\right)^{\frac{r-1}{r}} \\
&= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{p-1}{p}\right)^{\frac{p-1}{p}} (\ln(x+1))^{\frac{p+1}{p}} \left((x+1)^{\frac{p}{p-1}} - 1\right)^{\frac{p-1}{p}} \\
&\times \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\frac{q-1}{q}\right)^{\frac{q-1}{q}} (\ln(x+1))^{\frac{q+1}{q}} \left((x+1)^{\frac{q}{q-1}} - 1\right)^{\frac{q-1}{q}} \\
&\times \left(\frac{1}{r+1}\right)^{\frac{1}{r}} \left(\frac{r-1}{r}\right)^{\frac{r-1}{r}} (\ln(x+1))^{\frac{r+1}{r}} \left((x+1)^{\frac{r}{r-1}} - 1\right)^{\frac{r-1}{r}} \\
&= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{p-1}{p}\right)^{\frac{p-1}{p}} \left((x+1)^{\frac{p}{p-1}} - 1\right)^{\frac{p-1}{p}} \\
&\times \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\frac{q-1}{q}\right)^{\frac{q-1}{q}} \left((x+1)^{\frac{q}{q-1}} - 1\right)^{\frac{q-1}{q}} \\
&\times \left(\frac{1}{r+1}\right)^{\frac{1}{r}} \left(\frac{r-1}{r}\right)^{\frac{r-1}{r}} \left((x+1)^{\frac{r}{r-1}} - 1\right)^{\frac{r-1}{r}} \\
&\times (\ln(x+1))^{\frac{p+1}{p}+\frac{q+1}{q}+\frac{r+1}{r}}.
\end{aligned}
$$

The inequality casts into the form

$$
\begin{aligned}
|\ln(x+1)(x+1) - x|^3 \leq\; & \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{p-1}{p}\right)^{\frac{p-1}{p}} \left((x+1)^{\frac{p}{p-1}} - 1\right)^{\frac{p-1}{p}} \\
&\times \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\frac{q-1}{q}\right)^{\frac{q-1}{q}} \left((x+1)^{\frac{q}{q-1}} - 1\right)^{\frac{q-1}{q}} \\
&\times \left(\frac{1}{r+1}\right)^{\frac{1}{r}} \left(\frac{r-1}{r}\right)^{\frac{r-1}{r}} \left((x+1)^{\frac{r}{r-1}} - 1\right)^{\frac{r-1}{r}} \\
&\times (\ln(x+1))^{\frac{p+1}{p}+\frac{q+1}{q}+\frac{r+1}{r}}.
\end{aligned}
$$

Pick $p = q = r = 2$, and the above estimate becomes

$$
|\ln(x+1)(x+1) - x|^{\,3} \leq (\ln(x+1))^{\frac{9}{2}} \left(\frac{1}{6}\right)^{\frac{3}{2}} \left(x^2 + 2x\right)^{\frac{3}{2}}, \ \forall\ x\ \in [0, +\infty[.
$$

□

5. Proof of Theorem 6

Proof. Consider $f(t) = t, t \in (0, d(x)),\ d(x) = ln(x+1), x \in (0, +\infty)$. Then

$$\int_0^{d(x)} \frac{f'(t)}{1+t} dt = \left[\frac{f(t)}{1+t}\right]_0^{d(x)} - \int_0^{d(x)} f(t) \left(\frac{1}{1+t}\right)' dt$$
$$= \frac{f(d(x))}{1+d(x)} - f(0) + \int_0^{d(x)} f(t)(1+t)^{-2} dt$$
$$\leq \frac{f(d(x))}{1+d(x)} - f(0) + \left(\int_0^{d(x)} |f|^p dt\right)^{\frac{1}{p}} \left(\int_0^{d(x)} (1+t)^{-\frac{2p}{p-1}} dt\right)^{\frac{p-1}{p}}$$
$$= \frac{f(d(x))}{1+d(x)} - f(0) + \left(\int_0^{d(x)} |f|^p dt\right)^{\frac{1}{p}} \left(\frac{p-1}{p+1}\right)^{\frac{p-1}{p}} \left(1 - (d(x)+1)^{\frac{-(p+1)}{(p-1)}}\right)^{\frac{p-1}{p}}$$
$$= \frac{\ln(x+1)}{1+\ln(x+1)} + \left(\frac{p-1}{p+1}\right)^{\frac{p-1}{p}} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} (\ln(x+1))^{\frac{p+1}{p}} \left(1 - (\ln(x+1)+1)^{-\frac{(p+1)}{(p-1)}}\right)^{\frac{p-1}{p}}.$$

Consequently, the inequality reads

$$\ln(\ln(x+1)+1) \leq \frac{\ln(x+1)}{1+\ln(x+1)} + \left(\frac{p-1}{p+1}\right)^{\frac{p-1}{p}} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} (\ln(x+1))^{\frac{p+1}{p}} \left(1 - (\ln(x+1)+1)^{-\frac{(p+1)}{(p-1)}}\right)^{\frac{p-1}{p}}.$$

By picking $p = 3$, the desired result follows

$$\ln(\ln(x+1)+1) \leq \frac{\ln(x+1)}{1+\ln(x+1)} + \left(\frac{1}{16}\right)^{\frac{1}{3}} (\ln(x+1))^{\frac{4}{3}} \left(1 - (\ln(x+1)+1)^{-2}\right)^{\frac{2}{3}},\ \forall x \in [0, +\infty[.$$

□

6. Proof of Theorem 7

Proof. Consider the function $\lambda(x) = \frac{1}{2}\ln\left(\left(ax^2+1\right)\left(bx^2+1\right)\left(cx^2+1\right)\left(dx^2+1\right)\right),\ x \in [0,+\infty[,\ a,b,c,d\ \in]0,+\infty[$. Then, it follows

$$
\begin{aligned}
&\frac{1}{2}\ln\left(\left(ax^2+1\right)\left(bx^2+1\right)\left(cx^2+1\right)\left(dx^2+1\right)\right)\\
&=\ln\left(\sqrt{ax^2+1}\right)+\ln\left(\sqrt{bx^2+1}\right)+\ln\left(\sqrt{cx^2+1}\right)+\ln\left(\sqrt{dx^2+1}\right)\\
&=\int_0^x \frac{1}{\sqrt{at^2+1}}\ \frac{a\ t}{\sqrt{at^2+1}}dt+\int_0^x \frac{1}{\sqrt{bt^2+1}}\ \frac{b\ t}{\sqrt{bt^2+1}}dt\\
&+\int_0^x \frac{1}{\sqrt{ct^2+1}}\ \frac{c\ t}{\sqrt{ct^2+1}}dt+\int_0^x \frac{1}{\sqrt{dt^2+1}}\ \frac{d\ t}{\sqrt{dt^2+1}}dt\\
&\leq\left(\int_0^x\left(\frac{1}{\sqrt{at^2+1}}\right)^2dt\right)^{\frac{1}{2}}\left(\int_0^x\left(\frac{a\ t}{\sqrt{at^2+1}}\right)^2dt\right)^{\frac{1}{2}}\\
&+\left(\int_0^x\left(\frac{1}{\sqrt{bt^2+1}}\right)^2dt\right)^{\frac{1}{2}}\left(\int_0^x\left(\frac{b\ t}{\sqrt{bt^2+1}}\right)^2dt\right)^{\frac{1}{2}}\\
&+\left(\int_0^x\left(\frac{1}{\sqrt{ct^2+1}}\right)^2dt\right)^{\frac{1}{2}}\left(\int_0^x\left(\frac{c\ t}{\sqrt{ct^2+1}}\right)^2dt\right)^{\frac{1}{2}}\\
&+\left(\int_0^x\left(\frac{1}{\sqrt{d\ t^2+1}}\right)^2dt\right)^{\frac{1}{2}}\left(\int_0^x\left(\frac{d\ t}{\sqrt{dt^2+1}}\right)^2dt\right)^{\frac{1}{2}}\\
&=a^{\frac{-1}{4}}\sqrt{\arctan\left(\sqrt{a}x\right)}\ \sqrt{ax-a^{\frac{1}{2}}\arctan\left(\sqrt{a}x\right)}\\
&+b^{\frac{-1}{4}}\sqrt{\arctan\left(\sqrt{b}x\right)}\ \sqrt{bx-b^{\frac{1}{2}}\arctan\left(\sqrt{b}x\right)}\\
&+c^{\frac{-1}{4}}\sqrt{\arctan\left(\sqrt{c}x\right)}\ \sqrt{cx-c^{\frac{1}{2}}\arctan\left(\sqrt{c}x\right)}\\
&+d^{\frac{-1}{4}}\sqrt{\arctan\left(\sqrt{d}x\right)}\ \sqrt{dx-d^{\frac{1}{2}}\arctan\left(\sqrt{d}\ x\right)}.
\end{aligned}
$$

By making the choice for the coefficients to be $a=2, b=3, c=5, d=7$, the final estimate is obtained

$$
\begin{aligned}
&\frac{1}{2}\ln\left(\left(2x^2+1\right)\left(3x^2+1\right)\left(5x^2+1\right)\left(7x^2+1\right)\right)\\
&\leq 2^{\frac{-1}{4}}\sqrt{\arctan\left(\sqrt{2}x\right)}\ \sqrt{2x-2^{\frac{1}{2}}\arctan\left(\sqrt{2}x\right)}\\
&+3^{\frac{-1}{4}}\sqrt{\arctan\left(\sqrt{3}x\right)}\ \sqrt{3x-3^{\frac{1}{2}}\arctan\left(\sqrt{3}x\right)}\\
&+5^{\frac{-1}{4}}\sqrt{\arctan\left(\sqrt{5}x\right)}\ \sqrt{5x-5^{\frac{1}{2}}\arctan\left(\sqrt{5}x\right)}\\
&+7^{\frac{-1}{4}}\sqrt{\arctan\left(\sqrt{7}x\right)}\ \sqrt{7x-7^{\frac{1}{2}}\arctan\left(\sqrt{7}x\right)}.
\end{aligned}
$$

□

7. Proof of Theorem 8

Proof. Let $f(t) = \arctan(t)$, $g(t) = t$, $t \in (0, x)$.Taking into account that both functions are monotonically increasing, it follows

$$\int_0^x f(t)dt \int_0^x g(t)dt \leq x \int_0^x f(t)g(t)dt \quad \textit{Chebyshev integral inequality}$$

$$\Rightarrow \frac{x^3}{2}\arctan(x) - \frac{x^2}{4}\ln\left(x^2+1\right) \leq \frac{x^3}{2}\arctan(x) - \frac{x^2}{2} + \frac{x}{2}\arctan(x)$$

$$\Rightarrow \frac{x^2}{2} - \frac{x}{2}\arctan(x) \leq \frac{x^2}{4}\ln\left(x^2+1\right)$$

$$\Rightarrow 2x(x - \arctan(x)) \leq x^2\ln\left(x^2+1\right)$$

and the proof is complete. □

8. Graphical evidence

In this section, inequalities are also verified using graphs. These illustrations confirm the rigorous proofs developed in the first section. All the graphs follow.

9. Conclusions and remarks

In this work, inequalities involving logarithmic functions have been derived with the help of integral-type inequalities. The inequalities are nontrivial and serve as a good enrichment of the literature of mathematical inequalities. All the estimates have been derived using purely functional inequalities and not relying on monotonicity techniques. Here are the comments for each one of the inequalities:

- Inequality (1): The logarithmic function is bounded by the product of various other functions, and some of them are square rooted. More precisely, at the right hand side there is a combination of monomial, polynomial, exponential, logarithmic and arctangent functions. Such inequality is novel in the literature. The inequality is sharp for small values of x and then becomes less sharp as x grows larger.
- Inequality (2): This is a sharp inequality for the logarithmic function, and the estimate is simpler compared to the one proposed by Chesneau-Bagul [3] where there is the arctan function at the right hand side of the inequality. Chesneau-Bagul bound is sharper than this bound for small x but as x grows larger, the estimate is better than the Bagul-Chesneau estimate. A future work would focus on finding optimal Hölder exponents to make the bound more sharp.
- Inequality (3): The identity function is bounded by sum of terms where there is the logarithmic function raised to fractional power and multiplied by the argument into a fractional power minus a constant. There are two remarks to be made: 1. the inequality is sharp, 2. the right hand side of the inequality is an interesting function as the various combinations of logarithmic functions to a fractional power multiplied by linear terms to a fractional power give a linear approximation. The Hölder exponents can control the sharpness of the inequality.
- Inequality (4): The left hand side of the inequality is the product of logarithmic function times a linear function minus the identity function, all this in modulus. The right hand side has a slight resemblance to the right hand side of (3) but the constants are different. This is an interpolation inequality and it is sharp.
- Inequality (5): This is an interpolation inequality. The left hand side of the inequality has the function mentioned in (4) (left hand side) raised to the cubic power. The right hand side has a constant coming from Hölder integration, multiplied by the logarithmic function raised to a fractional power , then multiplied

by a polynomial function in a fractional power. This interpolation inequality is sharp, verified by the appropriate graph.

- Inequality (6):The left hand side is a nested logarithmic function bounded by the sum of two terms containing the logarithmic function. The inequality is sharp for small values of x and as x grows larger then the inequality becomes less sharp.
- Inequality (7): The left hand side of the inequality is a logarithmic function where the argument is a product of polynomials of degree two. This function is bounded by a function containing terms with arctangent functions which are square rooted. The inequality is sharp for small values of x and becomes less sharp as x becomes larger.
- Inequality (8): This inequality is derived using the Chebyshev integral inequality. The left hand side is a parabola multiplied by the logarithmic function. The right hand side consists of a linear function multipled by the difference between the identity and arctan function. This inequality could also be proved using monotonicity properties. A future work would include the derivation of sharper estimates and more novel involving logarithmic functions.

To conclude this work, the following statement could be made: New inequalities can arise by combining various integral inequalities, and these combinations could determine how sharp these are. Additionally, when using the Hölder inequality, finding the appropriate exponents to optimize the inequalities could be a potential type of research.

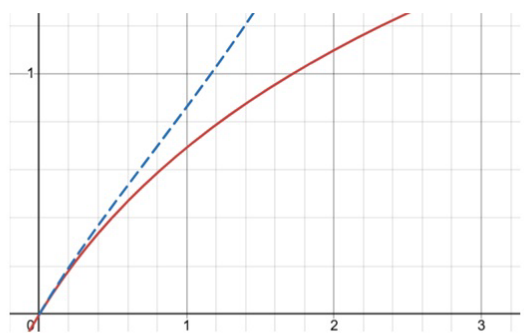

Figure 1. Graphical representation of estimate (1)

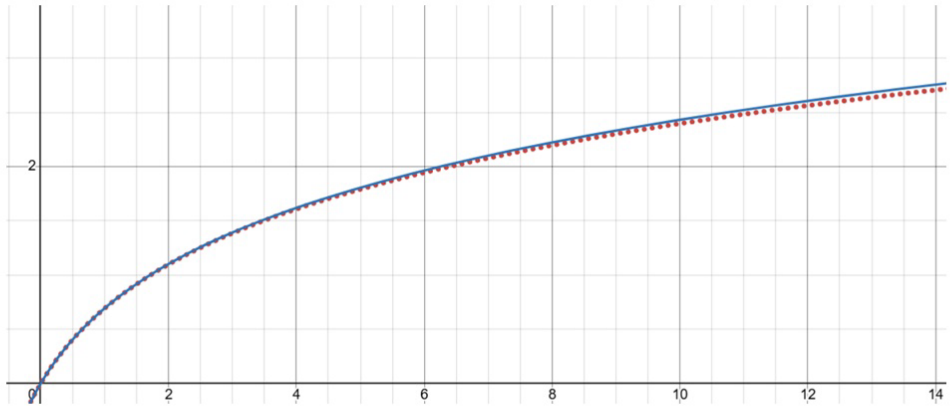

FIGURE 2. Graphical representation of estimate (2)

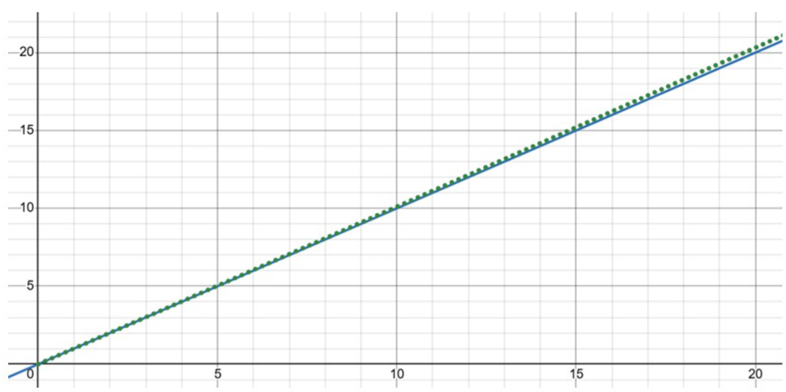

FIGURE 3. Graphical representation of estimate (3)

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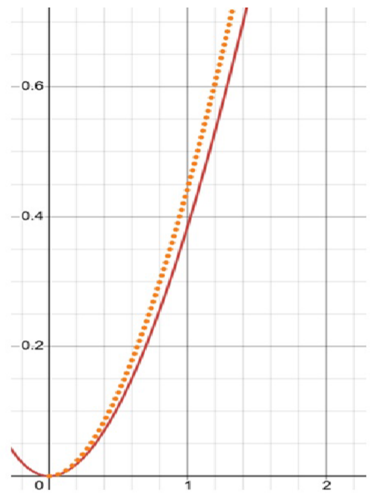

FIGURE 4. Graphical representation of estimate (4)

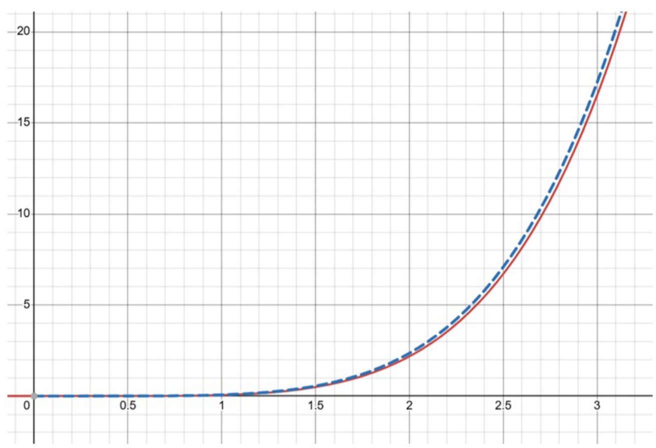

FIGURE 5. Graphical representation of estimate (5)

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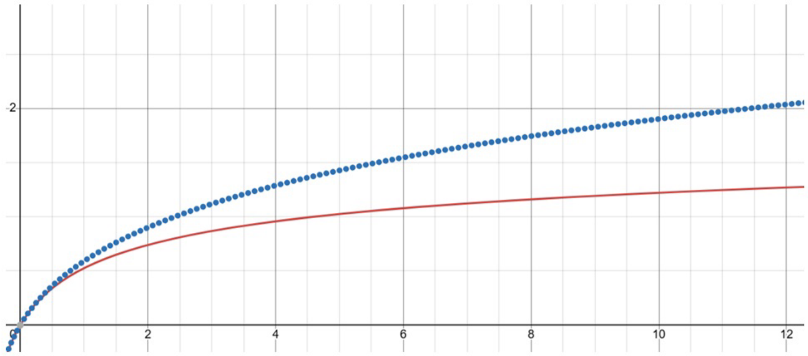

FIGURE 6. Graphical representation of estimate (6)

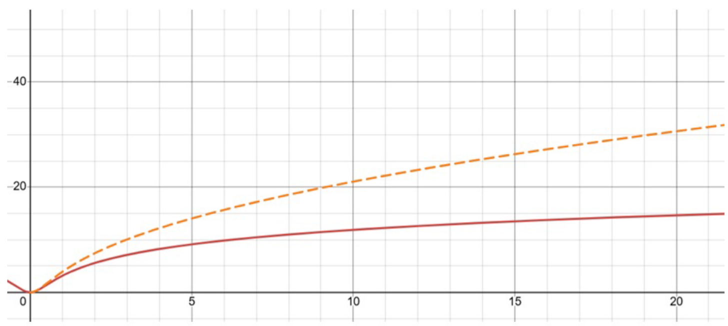

FIGURE 7. Graphical representation of estimate (7)

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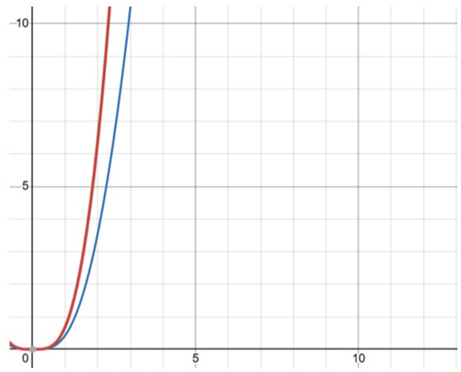

FIGURE 8. Graphical representation of estimate (8)

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